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Electromagneto Gas and Fluid Dynamics: An Analysis of Two-Dimensional Orthogonal and Constantly Inclined Flows

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ELECTROMAGNETO GAS AND FLUID DYNAMICS:

AN ANALYSIS OF TWO-DIMENSIONAL

ORTHOGONAL AND CONSTANTLY INCLINED FLOWS

by

Alka Chandna

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DEDICATION

My life has been enriched and made fuller by the presence of a good many people, and it is with much gratitude that I offer sincere and heartfelt thanks to all of them. However, I wish to dedicate this work to my beloved family, without whom this small achievement would have been both impossible and meaningless. Words fall short of adequately expressing the thanks which are due to my parents and to my brother and sisters, for the love and support which they have given to me, at all times, and in all of my endeavours. I only hope that I can be worthy of what they have given to me, by emulating their humanity and compassion in my own actions.

ABSTRACT

This thesis is devoted to a mathematical study of steady electromagneto fluid and gas dynamic flows. Steady plane flows are studied for the cases in which the magnetic field and velocity field are (i)orthogonal to one another, and (ii)at a constant angle of inclination to one another. Throughout the study, we consider, strictly, fluids having non-zero charge density.

Flows of inviscid compressible, inviscid incompressible and viscous incompressible electrically conducting fluids are considered in the case of orthogonal flows. Using as a starting point, the development of a system of equations governing a steady plane electro-magneto-gasdynamic flow, the following work is done for this case:

(i) Integrability conditions are determined for the unknown scalar functions, to establish their existence.

(ii) Employing complex variable techniques, the possible solutions for the flow in question are determined, taking the net formed by the streamlines and their

orthogonal trajectories to be isometric.

Flows of inviscid compressible electrically conducting fluids are considered in the case of constantly inclined flows. Again, the integrability conditions for the various unknown scalar functions involved in the system of governing equations are found. As well, a study of circulation of flow is undertaken in this section of the thesis.

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CHAPTER I

INTRODUCTION

1.1 HISTORICAL SKETCH

Electro-magneto-fluid-dynamics (EMFD) is the study of the motion of electrically conducting fluids in the presence of both electric and magnetic fields. The interaction of the elctromagnetic field and the ionized fluid gives rise to mechanical forces which alter the fluid flow. Many new phenomena occur due to the interaction of the fluid dynamic and elctromagnetic forces.

The mathematical study of EMFD is concerned primarily with the partial differential equations which arise from the well known physical conservation laws. A system of sixteen equations in as many unknowns, of which ten are electromagnetic, three are dynamic and three are thermodynamic, governs the dynamics of flow. Theoretical analysis of EMFD flows is largely prohibited by the complexity of the governing mathematical system of partial differential equations.

Magneto-fluid-dynamics (MFD) is the result of an additional assumption that the energy in the elctric field is much smaller than in the magnetic field so that the elctromagnetic quantities are expressed in terms of the magnetic field. A vast amount of research has been carried out on the motion of electrically conducting fluids, using MGD approximations, since Alfven's classic work [1]. Mathematical complexity of the MFD phenomenon induced many of the researchers to adopt a rather useful alternate technique of investigating special classes of flows such as aligned or orthogonal flows. These special classes of flows in MFD yielded various solvable second order mathematical structures. However, not much work seems to have been done in EMFD.

The general problem of EMFD is quite complex. Kingston and Power, in 1968, [2] initiated two-dimensional aligned flows and gave an elegant analysis of its dynamics by employing complex variable techniques. Considering compressible and incompressible fluids with finite electric conductivity, they established that if charge density is not identically zero in a flow region, then the magnetic field must be irrotational everywhere. However, the results of this work were largely restricted to the special case where the magnetic field is a constant multiple of the mass flux.

In 1971, Chandna and Holmes [3] extended the results of Kingston and Power for axially symmetric flows. They proved that for steady axially symmetric aligned flows, of conducting fluids of finite conductivity, either the charge density is zero or the magnetic field is irrotational.

Chandna, Toews and Prabaharan [4], in 1981, studied incompressible aligned steady plane flows with non-zero charge density, without assuming proportionality between the magnetic field and the aligned velocity field. They reduced the system of governing equations to the linear momentum equation, the continuity equation, and two integrability conditions for the velocity field which ensure the existence of the charge density distribution and the magnetic field. In this work, the authors introduced additional kinematical assumptions to obtain special familites of velocity fields. In 1984, Yin [5] succeeded in completely analysing incompressible aligned steady plane flows with non-zero charge density, without assuming proportionality between the magnetic field and the velocity field, and without introducing special kinematical assumptions. Yin concluded that all such flows are irrotational motions.

Barron, Chandna and Prabaharan [6], in 1984, examined the full EMFD equations for the steady plane flow of a viscous, incompressible fluid of finite electrical conductivity.

Orthogonal flows were taken. That is, the magnetic field was assumed to lie in the plane of the flow and to be everywhere perpendicular to the velocity field. For non-zero electric charge density, a complete classification of such flows was achieved and the corresponding solution of the EMFD equations was obtained.

In 1985, Chandna and Rudraiah [7] studied steady EMGD aligned flows when the charge density is not identically zero. The authors considered general aligned flows and a general equation of state for the gas was taken.

Chandna, Toews and Rudraiah [8], in 1986 studied steady plane aligned EMFD flows for three different classes of fluids: (i) viscous, incompressible fluids with non-zero charge density, (ii) viscous, compressible fluids with non-zero charge density and (iii) inviscid, adiabatic, compressible fluids with zero charge density.

CHAPTER II

PRELIMINARIES

2.1 *BASIC EQUATIONS*

The general problem of electro-magneto-gas or fluid dynamics without employing any MHD approximation is quite complex. The aim of this work is to study, without any MHD approximation, some:

(i) plane inviscid compressible EMGD orthogonal flows,

(ii) plane inviscid incompressible EMFD orthogonal flows,

(iii) plane viscous incompressible EMFD orthogonal flows, and

(iv) plane inviscid compressible EMGD constantly inclined flows,

in the case where the charge density function is a non-zero function.

The unsteady inviscid three dimensional electro-magneto-gas dynamic flow is governed by the system: [10]

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0 \tag{1}$$

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\circ\nabla)\mathbf{v}\right] = -\mathrm{grad}\ p + \mu \mathbf{J}\times\mathbf{H} + q\mathbf{E} \tag{2}$$

$$\mathrm{curl}\ \mathbf{H} = \mathbf{J} + \varepsilon\,\frac{\partial \mathbf{E}}{\partial t} \tag{3}$$

$$\mathrm{curl}\ \mathbf{E} = -\mu\,\frac{\partial \mathbf{H}}{\partial t} \tag{4}$$

$$\text{div } \mathbf{E} = \frac{q}{\varepsilon} \tag{5}$$

$$\mathbf{J} = \mathbf{I} + q\mathbf{v} = \sigma \, [\mathbf{E} + \mu \mathbf{v}\times\mathbf{H}] + q\mathbf{v}] \tag{6}$$

$$\frac{\partial}{\partial t} [\rho e + 1/2 \, \rho\mathbf{v}\cdot\mathbf{v}] + \text{div}[(\rho e + 1/2 \, \rho \, \mathbf{v}\cdot\mathbf{v})\mathbf{v}] = -\text{div}(\rho\mathbf{v}) + \mathbf{E}\cdot\mathbf{J} + \text{div}(k \text{ grad } T) \tag{7}$$

$$p = p \, (\rho, T) \tag{8}$$

This is a system of sixteen fundamental equations in sixteen unknowns $\mathbf{v}$ (the velocity field), $\mathbf{H}$ (the magnetic field), $\mathbf{E}$ (the electric field), $\mathbf{J}$ (the current density function), q(the charge density function) and three thermodynamic variables p, ρ and T. In this system, $\sigma, \mu, \varepsilon, k, e$ are respectively the constant electric conductivity, the constant magnetic permeability, the constant permitivity, the constant thermal conductivity, and the specific, internal energy function . In the case of an ideal gas, the specific internal energy functio e is a function of temperature only. In the case of ideal polytropic gases, $e = c_v T$. The magnetic field $\mathbf{H}$ satisfies an addititional equation

$$\text{div } \mathbf{H} = 0 \tag{9}$$

expressing the absence of magnetic poles in the flow.

Taking the flow under consideration to be steady, we can eliminate the terms involving time dependence from the above system. The system of equations governing this flow is:

$$\operatorname{div}(\rho \mathbf{v}) = 0 \tag{10}$$

$$\rho[\mathbf{v}\circ\nabla]\mathbf{v} = -\operatorname{grad} p + \mu(\mathbf{J}\times\mathbf{H}) + q\mathbf{E} \tag{11}$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J} \tag{12}$$

$$\operatorname{curl} \mathbf{E} = \mathbf{0} \tag{13}$$

$$\operatorname{div} \mathbf{E} = \frac{q}{\varepsilon} \tag{14}$$

$$\mathbf{J} = \mathbf{I} + q\mathbf{v} = \sigma[\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}] + q\mathbf{v} \tag{15}$$

$$\operatorname{div}(\rho e + 1/2\rho(\mathbf{v}\circ\mathbf{v})\mathbf{v}) = -\operatorname{div}(p\mathbf{v}) + \mathbf{E}\circ\mathbf{J} + \operatorname{div}(k \operatorname{grad} T) \tag{16}$$

$$p = p(\rho, T) \tag{17}$$

From equations (12) and (15), we obtain the conduction current $\mathbf{I}$ and the electric field $\mathbf{E}$ as given by:

$$\mathbf{I} = \nabla \times \mathbf{H} - q\mathbf{v} \tag{18}$$

$$\mathbf{E} = \frac{1}{\sigma}\,[\nabla \times \mathbf{H} - q\mathbf{v}] - \mu\mathbf{v}\times\mathbf{H} \tag{19}$$

Eliminating $\mathbf{E},\mathbf{J}$ from equations (11),(13) and (14), by using their expressions from equations (19) and (12), we get

$$\rho[\mathbf{v}\circ\underline{\nabla}]\mathbf{v} = -\operatorname{grad} p + \mu(\operatorname{curl} \mathbf{H})\times\mathbf{H} + \frac{q}{\sigma}\,(\operatorname{curl} \mathbf{H} - q\mathbf{v}) - q\mu\mathbf{v}\times\mathbf{H} \tag{20}$$

$$\operatorname{curl}[\operatorname{curl}\mathbf{H} - q\mathbf{v} - \mu\sigma\mathbf{v}\times\mathbf{H}] = 0 \tag{21}$$

$$\operatorname{div}[q\mathbf{v} + \mu\sigma\mathbf{v}\times\mathbf{H}] + \frac{q\sigma}{\epsilon} = 0 \tag{22}$$

Hence, our new system of governing equations is:

$$\operatorname{div}(\rho\mathbf{v}) = 0$$

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\text{grad } p + \mu(\text{curl } \mathbf{H})\times\mathbf{H} + \frac{q}{\sigma}(\text{curl}\mathbf{H} - q\mathbf{v}) - \mu q\mathbf{v}\times\mathbf{H}$$

$$\text{curl}[\text{curl}\mathbf{H} - q\mathbf{v} - \mu\sigma\mathbf{v}\times\mathbf{H}] = \mathbf{0}$$

$$\text{div}[q\mathbf{v} + \mu\sigma(\mathbf{v}\times\mathbf{H})] + \frac{q\sigma}{\varepsilon} = 0$$

along with the energy equation and the state equation for the fluid. This is a system of ten equations in the ten unknowns: $\mathbf{v}, \mathbf{H}, \rho, T, q$ and p. Having solved this system, we can find the current density function $\mathbf{J}$, and the electric field $\mathbf{E}$, from equations (12) and (19), respectively.

Plane Flows: We investigate steady plane flows defined as those flows in which the velocity field $\mathbf{v}$ and the magnetic field $\mathbf{H}$ lie in a plane defined by the rectangular coordinates x, y and all the flow variables are functions of x and y only.

We initiate the decomposition of the vector equations (20) and (21) into their vector components in the flow plane and their vector components perpendicular to the flow plane. This decomposition yields the following four equations:

$$\rho(\mathbf{v}\cdot\underline{\nabla})\,\mathbf{v} = -\text{grad } p + \mu(\text{curl } \mathbf{H})\times\mathbf{H} - \frac{q^2}{\sigma}\,\mathbf{v} \tag{23}$$

$$q[\text{curl } \mathbf{H} - \mu\sigma\mathbf{v}\times\mathbf{H}] = \mathbf{0} \tag{24}$$

$$\text{curl}[\text{curl } \mathbf{H} - \mu\sigma\mathbf{v}\times\mathbf{H}] = \mathbf{0} \tag{25}$$

$$\text{curl}[q\mathbf{v}] = \mathbf{0} \tag{26}$$

where equations (23), (24) result from (20) and equations

(25),(26) result from (21).

Employing the assumption that flow variables are functions of x,y only, equation (22) reduces to

$$\operatorname{div}(q\mathbf{v}) + \frac{\sigma q}{\epsilon} = 0 \tag{27}$$

From equation (24) and the above analysis, we have the following theorems:

Theorem 1. If the charge density $q(x,y)$ is a non-zero function in a steady plane electro-magneto-gasdynamic flow having the magnetic field $\mathbf{H}$ in the flow plane, then the flow is governed by the system:

$$\operatorname{div}(\rho\mathbf{v}) = 0 \tag{10}$$

$$\rho(\mathbf{v}\circ\underline{\nabla})\mathbf{v} = -\operatorname{grad} p + \mu(\underline{\nabla}\times\mathbf{H})\times\mathbf{H} - \frac{q^2}{\sigma}\mathbf{v} \tag{23}$$

$$\operatorname{curl}\mathbf{H} = \mu\sigma(\mathbf{v}\times\mathbf{H}) \tag{24}$$

$$\operatorname{curl}(q\mathbf{v}) = \mathbf{0} \tag{26}$$

$$\operatorname{div}(q\mathbf{v}) + \frac{q\sigma}{\epsilon} = 0 \tag{27}$$

along with the equations of energy and state. This is a system of eight scalar equations in eight unknown functions $\rho(x,y)$, $p(x,y)$, $T(x,y)$, $q(x,y)$, $\mathbf{v}=(u(x,y),v(x,y))$ and $\mathbf{H} = (H_1(x,y),H_2(x,y))$. The magnetic field $\mathbf{H}$ also satisfies

$$\operatorname{div}\mathbf{H} = 0. \tag{9}$$

Theorem 2. If the charge density $q(x,y)$ is identically zero in a steady plane electro-magneto-gasdynamic flow having the magnetic field $\mathbf{H}$ in the flow plane, then the flow is governed by the system:

$$\operatorname{div}(\rho \mathbf{v}) = 0 \tag{10}$$

$$\rho(\mathbf{v}\cdot\underline{\nabla})\mathbf{v} = -\underline{\nabla}\, p + \mu(\underline{\nabla}\times\mathbf{H})\times\mathbf{H} \tag{23}$$

$$\operatorname{curl}[\operatorname{curl}\mathbf{H} - \mu\sigma(\mathbf{v}\times\mathbf{H})] = \mathbf{0} \tag{25}$$

along with the equations of state and energy. This is a system of seven scalar equations in seven unknown functions $\rho(x,y), p(x,y), T(x,y), \mathbf{v}=(u(x,y),v(x,y))$ and $\mathbf{H} = (H_1(x,y), H_2(x,y))$.

The magnetic field $\mathbf{H}$ also satisfies

$$\operatorname{div}\mathbf{H} = 0. \tag{9}$$

CHAPTER III

INVISCID COMPRESSIBLE EMGD ORTHOGONAL FLOWS WITH NON-ZERO CHARGE DENSITY

3.1 *INTEGRABILITY CONDITIONS FOR ORTHOGONAL FLOWS WITH NON-ZERO CHARGE DENSITY*

The work carried out in this section employs a well established fluid dynamical technique of establishing integrability conditions for scalar fields. Ratip Berker [11] has used and thoroughly documented the application of integrability equations for incompressible and compressible fluid flows. As well, Berker, Ozoklav [12] and Chandna [13] used this approach for ordinary gas dynamics. An integrability condition is the necessary and sufficient condition which establishes the existence of a scalar function, when the velocity field is known. In the present work, an integrability condition cannot be found for the pressure function, but an integrability equation is found for the product of the scalar functions $f(x,y)$ and $\rho(x,y)$, as well as for $q(x,y)$, where $f(x,y)$ is a scalar function which shall appear shortly in the expression for **H**.

An investigation of inviscid compressible EMGD orthogonal flows is undertaken in this section. Plane orthogonal flows are qualitatively understood as those flows in which the velocity field $\mathbf{v}$ and the magnetic field $\mathbf{H}$ lie in the (x,y)-plane, and furthermore, $\mathbf{v}$ and $\mathbf{H}$ are perpendicular to one another. Formally, we define plane orthogonal flows as those flows for which the magnetic field $\mathbf{H}$ and the velocity field $\mathbf{v}$ satisfy:

$$\mathbf{H} = \mathbf{k} \times f(x,y)\ \rho(x,y)\ \mathbf{v} \tag{28}$$

where $f(x,y)$ is some scalar function and $\mathbf{k}=(0,0,1)$.

Using (28) in Theorem 1, and taking inviscid compressible fluid, the orthogonal steady plane electro-magneto-gasdynamic flow with non-zero charge density is governed by:

$$\operatorname{div}(\rho\mathbf{v}) = 0 \tag{29}$$

$$\rho(\mathbf{v}\cdot\operatorname{grad})\mathbf{v} = -\operatorname{grad} p + \mu[\operatorname{curl}(\mathbf{k}\times f\rho\mathbf{v})]\times(\mathbf{k}\times f\rho\mathbf{v}) - \frac{q^2}{\sigma}\mathbf{v} \tag{30}$$

$$\operatorname{curl}[\mathbf{k}\times f\rho\mathbf{v}] = \mu\sigma[\mathbf{v}\times(\mathbf{k}\times f\rho\mathbf{v})] \tag{31}$$

$$\operatorname{curl}(q\mathbf{v}) = \mathbf{0} \tag{32}$$

$$\operatorname{div}(q\mathbf{v}) + \frac{q\sigma}{\varepsilon} = 0 \tag{33}$$

along with the equations of energy and state. Employing (28) in (9), we get:

$$\operatorname{div}[\mathbf{k}\times f\rho\mathbf{v}] = 0 \tag{34}$$

Equations(31) and (34) can respectively be written as:

$$\operatorname{grad}(\ln f\rho) = \frac{(\operatorname{curl}\mathbf{v})\times\mathbf{v} + (\mathbf{v}\circ\operatorname{grad}(\ln f\rho))\mathbf{v}}{\mathbf{v}\circ\mathbf{v}} \tag{35}$$

$$\operatorname{div}(\mathbf{v}) + \operatorname{grad}(\ln f\rho)\circ\mathbf{v} = \mu\sigma(\mathbf{v}\circ\mathbf{v}) \tag{36}$$

From equations (35) and (36), we obtain the following expression for grad(ln $f\rho$):

$$\mathrm{grad}(\ln f\rho) = \frac{(\mathrm{curl}\ \mathbf{v})\times\mathbf{v} - (\mathrm{div}\ \mathbf{v})\mathbf{v}}{\mathbf{v}\circ\mathbf{v}} + \mu\sigma\mathbf{v} \qquad (37)$$

Taking the curl of equation (37) it follows that:

$$\mathrm{curl}\left[\frac{(\mathrm{curl}\ \mathbf{v})\times\mathbf{v} - (\mathrm{div}\ \mathbf{v})\mathbf{v}}{\mathbf{v}\circ\mathbf{v}} + \mu\sigma\mathbf{v}\right] = 0 \qquad (38)$$

which is the integrability condition for the function $f\rho$. This is the first integrability condition. The second one is the integrability condition for q and we obtain it from equations (32) and (33). These equations can be rewritten as:

$$\mathrm{grad}(\ln q)\cdot\mathbf{v} + (\mathrm{div}\ \mathbf{v} + \frac{\sigma}{\varepsilon}) = 0 \qquad (39)$$

$$\mathrm{grad}(\ln q) = \frac{(\mathrm{curl}\ \mathbf{v})\times\mathbf{v} + (\mathrm{grad}(\ln q)\cdot\mathbf{v})\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} \qquad (40)$$

From equations (39) and (40), we obtain the following expression for grad(lnq)

$$\mathrm{grad}(\ln q) = \frac{(\mathrm{curl}\ \mathbf{v})\times\mathbf{v} - (\mathrm{div}\ \mathbf{v} + \sigma/\varepsilon)\mathbf{v}}{\mathbf{v}\circ\mathbf{v}} \qquad (41)$$

Taking the curl of equation (41), we get the integrability condition for q which is given by:

$$\mathrm{curl}\left[\frac{(\mathrm{curl}\ \mathbf{v})\times\mathbf{v} - (\mathrm{div}\ \mathbf{v} + \sigma/\varepsilon)\mathbf{v}}{\mathbf{v}\circ\mathbf{v}}\right] = 0 \qquad (42)$$

$\mathbf{v} \circ \mathbf{v}$

It follows, from equations (38) and (42), that for plane orthogonal flows, the vector field $\mathbf{v}$ satisfies

$$\mathrm{curl}\left[\left(\varepsilon\mu + \frac{1}{\mathbf{v}\circ\mathbf{v}}\right)\mathbf{v}\right] = \mathbf{0} \tag{43}$$

We let $\mathbf{v} = u\mathbf{e}_x + v\mathbf{e}_y = v_1\mathbf{e}_r + v_2\mathbf{e}_\theta$ express the components of the velocity field in rectangular and polar coordinates respectively. Equation (43), then yields:

$$\frac{\partial}{\partial x}\left[\left(\varepsilon\mu + \frac{1}{u^2+v^2}\right)v\right] - \frac{\partial}{\partial y}\left[\left(\varepsilon\mu + \frac{1}{u^2+v^2}\right)u\right] = 0 \tag{44}$$

$$\frac{\partial}{\partial r}\left[\left(\varepsilon\mu + \frac{1}{v_1^2+v_2^2}\right)rv_2\right] - \frac{\partial}{\partial \vartheta}\left[\left(\varepsilon\mu + \frac{1}{v_1^2+v_2^2}\right)v_1\right] = 0 \tag{45}$$

Equation (44) implies the existence of a function $\phi(x,y)$, and equation (45) implies the existence of a function $\Phi(r,\vartheta)$, such that:

$$\frac{\partial\phi}{\partial x} = \left(\varepsilon\mu + \frac{1}{u^2+v^2}\right)u \tag{46a}$$

$$\frac{\partial\phi}{\partial y} = \left(\varepsilon\mu + \frac{1}{u^2+v^2}\right)v \tag{46b}$$

and

$$\frac{\partial\Phi}{\partial r} = \left(\varepsilon\mu + \frac{1}{v_1^2+v_2^2}\right)v_1 \tag{47a}$$

$$\frac{\partial\Phi}{\partial\theta} = \left(\varepsilon\mu + \frac{1}{v_1^2+v_2^2}\right)rv_2 \tag{47b}$$

$\partial\partial$ $\qquad v_1^2 + v_2^2$

From the above equations, we have:

$$\frac{u}{(u^2 + v^2)^{1/2}} = \frac{\frac{\partial\phi}{\partial x}}{\left[(\partial\phi/\partial x)^2 + (d\phi/\partial y)^2\right]^{1/2}} \tag{48a}$$

$$\frac{v}{(u^2 + v^2)^{1/2}} = \frac{\frac{\partial\phi}{\partial y}}{\left[(\partial\phi/\partial x)^2 + (d\phi/\partial y)^2\right]^{1/2}} \tag{48b}$$

and

$$\frac{v_1}{(v_1^2 + v_2^2)^{1/2}} = \frac{r\,\frac{\partial\Phi}{\partial r}}{\left[r^2\,\Phi_r^2 + \Phi_\theta^2\right]^{1/2}} \tag{49a}$$

$$\frac{v_2}{(v_1^2 + v_2^2)^{1/2}} = \frac{\frac{\partial\Phi}{\partial\theta}}{\left[r^2\,\Phi_r^2 + \Phi_\theta^2\right]^{1/2}} \tag{49b}$$

The determination of exact solutions for the flows governed by equations (29) to (33) requires the determination of the velocity field **v which** satisfies equations (38), (42) and (43) in the first step. Once we know the velocity field, we determine $q(x,y)$, $f(x,y)\rho(x,y)$, $\mathbf{H}(x,y)$, and $\rho(x,y)$ using equations (41), (37), (28) and (29). The pressure function, $p(x,y)$ is determined from the linear momentum equations, provided $p_{xy}=p_{yx}$ holds true.

We now consider a number of examples.

EXAMPLE I (Straight Parallel Flows)

Taking $ax + by = \text{const.}$ and $ay-bx = \text{const.}$ to be the families of streamlines and their orthogonal trajectories, respectively, we have

$$\phi = F(ay-bx) \qquad (50)$$

Using this expression for ϕ in equation (48), we get

$$\frac{u}{(u^2 + v^2)^{1/2}} = \frac{-b\ F'(ay-bx)}{(a^2+b^2)^{1/2}F'(ay-bx)}$$

$$\frac{v}{(u^2 + v^2)^{1/2}} = \frac{a\ F'(ay-bx)}{(a^2+b^2)^{1/2}F'(ay-bx)}$$

Thus,

$$\mathbf{v} = (u, v) = \frac{Q}{(a^2+b^2)^{1/2}}(-b\ \mathbf{e}_x + a\ \mathbf{e}_y) \qquad (51)$$

where $Q^2 = u^2 + v^2$.

Employing equation (51) in the integrability condition for $f\rho$, equation (38), we obtain:

$$\text{curl}\ \{\ [(-Q_x/Q) - ((b\mu\sigma Q)/(a^2+b^2)^{1/2})]\ \mathbf{i} + [(-Q_y/Q) + ((b\mu\sigma Q)/(a^2+b^2)^{1/2})]\ \mathbf{j} = \mathbf{0}$$

This yields the partial differential equation

$$a\ Q_x + b\ Q_y = 0$$

which has the general solution

$$Q = G(bx-ay) \qquad (52)$$

where G is an arbitrary function of its argument.

Thus, the velocity field can be expressed as

$$v = \frac{G}{(a^2+b^2)^{1/2}} (-b\, i + a\, j) \tag{53}$$

Employing equation (53) in equations (37) and (40) yields:

$$[\ln(f\rho G)]_x = \frac{-b\mu\sigma G}{(a^2+b^2)^{1/2}} \tag{54}$$

$$[\ln(f\rho G)]_y = \frac{a\mu\sigma G}{(a^2+b^2)^{1/2}} \tag{55}$$

$$[\ln(qG)]_x = \frac{\sigma b}{\varepsilon G(a^2+b^2)^{1/2}} \tag{56}$$

$$[\ln(qG)]_y = \frac{-\sigma a}{\varepsilon G(a^2+b^2)^{1/2}} \tag{57}$$

Integration of equations (54) to (57) yields

$$f\rho = \frac{1}{G} \exp [\ \mu\sigma\ (a^2+b^2)^{-1/2} \int(aGdy-bGdx)] \tag{58}$$

and

$$q = \frac{1}{G} \exp\ [\ \varepsilon^{-1}\sigma\ (a^2+b^2)^{-1/2} \int(bG^{-1}dy-aG^{-1}dx)] \tag{59}$$

Employing equation (53) in the continuity equation, (20), we obtain the partial differential equation

$$-b\ (G\rho)_x + a\ (G\rho)_y = 0$$

The above equation has the general solution

$$G\rho = K(ax+by)$$

where K is an arbitrary function of its argument.

Therefore,

$$\rho(x,y) = \frac{K(ax+by)}{G(bx-ay)} \tag{60}$$

Employing equation (60) in equation (58), we get

$$f(x,y) = \frac{1}{K(ax+by)} \exp [\ \mu\sigma\ (a^2+b^2)^{-1/2}\ \int(aGdy-bGdx)] \qquad \ldots\ldots (61)$$

Employing equations (60) and (61) in equation (28), we find the magnetic field to be

$$\mathbf{H} = (a^2+b^2)^{-1/2}\ [\ \mu\sigma\ (a^2+b^2)^{-1/2}\ \int(aGdy-bGdx)]\ (-ai+bj) \qquad \ldots\ldots (62)$$

Taking α = bx-ay and β = ax+by, we may rewrite equations (53),(59),(60),(61) as follows:

$$\mathbf{v} = (a^2+b^2)^{-1/2}\ G(\alpha)\ (-b\ i + a\ j) \qquad (63)$$

$$q = \frac{1}{G(\alpha)} \exp [\sigma\varepsilon^{-1}\ (a^2+b^2)^{-1/2}\ \int G^{-1}(\alpha)\ d\alpha\ ] \qquad (64)$$

$$\rho = \frac{K(\beta)}{G(\alpha)} \qquad (65)$$

$$f = \frac{1}{K(\beta)} \exp [-\mu\sigma\ \ (a^2+b^2)^{-1/2}\ \int G(\alpha)d\alpha] \qquad (66)$$

$$\mathbf{H} = (a^2+b^2)^{-1/2}\ \exp [-\mu\sigma\ (a^2+b^2)^{-1/2}\ \int G(\alpha)d\alpha]\ (-a,b) \qquad \ldots (67)$$

Furthermore, we express the linear momentum equations (30), taking α and β as the independent variables, and employ equations (63),(65) and (67), to obtain:

$$+bp_\alpha - ap_\beta = bKG' + \frac{\mu^2\sigma b\ G'(\alpha)}{(a^2+b^2)^{1/2}} \exp [\ -2\mu\sigma(a^2+b^2)^{-1/2} \int G(\alpha)d\alpha]$$

$$+ \frac{b}{(a^2+\ b^2)^{1/2}\sigma\ G(\alpha)} \exp [2\sigma\varepsilon^{-1}\ (a^2+b^2)^{-1/2}\ \int G^{-1}(\alpha)d\alpha]$$

..(68)

$$-ap_{\alpha}-bp_{\beta}=aKG' - \frac{\mu^2\sigma a\ G'(\alpha)}{(a^2+b^2)^{1/2}} \exp\ [-2\mu\sigma(a^2+b^2)^{-1/2}\int G(\alpha)d\alpha]$$

$$- \frac{a}{(a^2+b^2)^{1/2}\sigma\ G(\alpha)} \exp\ [2\sigma\varepsilon^{-1}\ (a^2+b^2)^{-1/2}\ \int G^{-1}(\alpha)d\alpha]$$

..(69)

Equations (68) and (69) yield the following expressions for p_{α} and p_{β}.

$$p_{\alpha} = K(\beta)G' + \frac{\mu^2\sigma\ \ G'(\alpha)}{(a^2+b^2)^{1/2}} \exp\ [-2\mu\sigma(a^2+b^2)^{-1/2}\ \int G(\alpha)]$$

$$+ \frac{1}{(a^2+b^2)^{1/2}\sigma\ G(\alpha)} \exp\ [2\sigma\varepsilon^{-1}\ (a^2+b^2)^{-1/2}\ \int G(\alpha)d\alpha]$$

(70)

$$p_{\beta} = 0 \qquad (71)$$

Applying the integrability condition $p_{\alpha\beta}=p_{\beta\alpha}$, we find that the arbitrary functions $G(\alpha)$ and $K(\beta)$ must satisfy: $K'(\beta)G'(\alpha) = 0$. This equation and the physical requirements imply that either:

i. $G(\alpha)$ is any non-zero constant with $K(\beta)$ as an arbitrary function,

OR ii. $K(\beta)$ is any non-zero constant with $G(\alpha)$ as an arbitrary non-zero function.

Solving for $p(\alpha)$, we get:

$$p(\alpha) = \begin{cases} (\varepsilon/2\sigma^2)\ \exp\ [(2\sigma/\varepsilon(a^2+b^2)^{1/2})G_0^{-1}\alpha + c_1] + c_2 & -(i) \\ \int p(\alpha)d\alpha & -(ii) \end{cases}$$

..............(72)

where G_0 is a non-zero constant, c_1 and c_2 are arbitrary constants and $p(\alpha)$ is a function of α given by equation (70), when $G(\alpha)$ is known.

EXAMPLE II (Radial Flow)

Taking θ = constant and r = constant to be the families of streamlines and their orthogonal trajectories, respectively, we have

$$\Phi = F(r) \tag{73}$$

Using this expression for Φ in equation (49), we get

$$v_1/|\mathbf{v}| = 1$$

$$v_2/|\mathbf{v}| = 0$$

where $|\mathbf{v}| = (v_1^2 + v_2^2)^{1/2}$.

Thus,

$$\mathbf{v} = (v_1, v_2) = Q\ \mathbf{e}_r \tag{74}$$

where $Q = |\mathbf{v}|$.

Employing equation (74) in equation (38), the integrability condition for $f\rho$, we obtain

$$\text{curl}\ \{\ [(-Q_r/Q) - (1/r) + \mu\sigma\ Q]\ \mathbf{e}_r + [(-Q_\theta/Qr)]\ \mathbf{e}_\theta = \mathbf{0}$$

This yields the partial differential equation

$$Q_\theta = 0$$

which has the general solution

$$Q = G(r) \tag{75}$$

Thus, the velocity field can be expressed as

$$\mathbf{v} = G(r)\ \mathbf{e}_r \tag{76}$$

Employing equation (76) in equations (37) and (40) yields

$$[\ln(f\rho G)]_r = -(1/r) + \mu\sigma\, G \tag{77}$$

$$[\ln(f\rho G)]_\theta = 0 \tag{78}$$

$$[\ln(qG)]_r = -(1/r) - (\sigma/\varepsilon G) \tag{79}$$

$$[\ln(qG)]_\theta = 0 \tag{80}$$

Integration of equations (77) to (80) yields

$$f\rho = (D_1/rG)\, \exp\, [\mu\sigma \int G\, dr\,] \tag{81}$$

and $$q = (D_2/rG)\, \exp\, [\, (-\sigma/\varepsilon) \int (1/G)\, dr\,] \tag{82}$$

where D_1 and D_2 are arbitrary constants.

Employing equation (76) in the continuity equation (29), we obtain the partial differential equation

$$[\ln(G\rho)]_r = -1/r$$

The above equation has the general solution

$$\rho\, G(r) = h(\theta)/r$$

where h is an arbitrary function of θ.

Therefore,

$$\rho(r,\theta) = \frac{h(\theta)}{rG(r)} \tag{83}$$

Employing equation (83) in equation (81), we get

$$f(r,\theta) = \frac{D_1}{h(\theta)}\, \exp\, [\, \mu\sigma \int G\, dr\,] \tag{84}$$

Employing equations (83) and (84) in equation (28), we find the magnetic field to be

$$\mathbf{H} = \frac{D_1}{r} \exp [\mu\sigma \int G \, dr] \, \mathbf{e}_\Theta \tag{85}$$

Using equations (76),(83) and (85) in the linear momentum equations (30), we get

$$\frac{\partial p}{\partial r} = -[h(\Theta)G'(r)]/r - \mu\sigma D_1^2 G(r) r^{-2} \exp[2\mu\sigma \int G(r)dr] + (D_2^2/\sigma r^2 G(r)) \exp[(-2\sigma/\varepsilon)\int G^{-1}(r)dr] \tag{86}$$

$$\frac{\partial p}{\partial \Theta} = 0 \tag{87}$$

Requiring $p_{r\Theta} = p_{\Theta r}$ yields

$$G'(r) \, h'(\Theta) = 0$$

Thus, we have the following possibilities:

i. $h'(\Theta)=0$ and $G'(r)=0$ [i.e., $h=h_0, G=G_0$, where h_0 and G_0 are arbitrary non-zero constants].

Then, the pressure function is found to be:

$$p(r) = D_5 + [(D_2^2 D_4 \varepsilon)/(2\sigma^2 r^2)] \exp[-2\sigma r/\varepsilon G_0] - [D_1^2 D_3 / 2r^2] \exp[2\mu\sigma G_0 r]$$

where D_3, D_4 and D_5 are arbitrary constants.

ii. $h'(\Theta)=0, G'(r)\neq 0$ [i.e., $h=h_0$, where h_0 is an arbitrary non-zero constant and $G(r)$ is an arbitrary function.]

The, the pressure function is found to be:

$$p(r) = -h_0 \int [G'(r)/r] dr - \mu\sigma D_1^2 \int \{G(r) r^{-1} \exp[2\mu\sigma \int G(r) dr]\} dr + (D_2^2/\sigma) \int \{r^{-2} G^{-1}(r) \exp[-2\sigma\varepsilon^{-1} \int G^{-1} dr]\} dr$$

iii. $G'(r)=0, h'(\theta)\neq 0$ [i.e., $G=G_0$ where G_0 is an arbitrary non-zero constant and $h(\theta)$ is an arbitrary function].

Then, the pressure function is found to be:

$$p(r) = D_8 + (D_2^2 D_7 \varepsilon/2\sigma^2 r^2) \exp[-2\sigma r/\varepsilon G_0] - (D_1^2 D_6/2r^2) \exp[2\mu\sigma G_0 r]$$

where D_6, D_7 and D_8 are arbitrary constants.

Hence, we have solved the radial flow problem, for orthogonal planar EMFD flow. We can use the state equation and the energy equation to find the temperature, T, and the unknown arbitrary function.

EXAMPLE III (Vortex Flows)

Taking r=constant and θ=constant to be the families of streamlines and their orthogonal trajectories, respectively, we have

$$\Phi = F(\theta) \tag{89}$$

Using this expression for Φ in equation (49), we get

$$\frac{v_1}{Q} = 0$$

$$\frac{v_2}{Q} = 1$$

where $Q=(v_1^2 + v_2^2)^{1/2}$

Thus

$$\mathbf{v} = (v_1, v_2) = 0\ e_r + Q\ e_\theta \tag{90}$$

Employing equation (90) in equation (38), the integrability condition for $f\rho$, we obtain

$$\mathrm{curl}\ [\ (-(\ln Q)_r-(1/r))\ \mathbf{e}_r + ((-1/r)(\ln Q)_\theta+\mu\sigma Q)\mathbf{e}_\theta\ ] = \mathbf{0}$$

This yields the partial differential equation

$$(rQ)_r = 0$$

which has the general solution

$$Q = \frac{G(\theta)}{r} \tag{91}$$

where G is an arbitrary function of its argument θ.

Thus, the velocity field can be expressed as

$$\mathbf{v} = 0\ \mathbf{e}_r + [G(\theta)/r]\ \mathbf{e}_\theta$$

Employing equation (90) in equation (42), the integrability condition for q, we obtain

$$\mathrm{curl}\ \{\ [-(\ln Q)_r-(1/r)]\mathbf{e}_r + [(-1/r)(\ln Q)_\theta - (\sigma/\varepsilon Q)]\mathbf{e}_\theta\ \} = \mathbf{0}$$

Therefore,

$$[\ln q]_r = -\ [\ln Q]_r - (1/r)$$

$$[\ln q]_\theta = -\ [\ln Q]_\theta - (\sigma r/\varepsilon Q)$$

Requiring $[\ln q]_{r\theta}=[\ln q]_{\theta r}$ yields

$$\frac{\partial}{\partial r}(r/Q) = 0$$

Thus,

$$(r/Q) = g(\theta)$$

where g is an arbitrary function of its argument, θ.

Employing equation (91) in (92), we obtain

$$r^2 = G(\theta)\ g(\theta) \tag{93}$$

Equation (93) is a contradiction. Therefore, vortex flow pattern for orthogonal inviscid EMGD plane steady flows is not possible.

EXAMPLE IV (Spiral Flows)

Taking re^{θ}=constant and $re^{-\theta}$=constant to be the families of streamlines and their orthogonal trajectories, respectively, we have

$$\Phi = F(re^{-\theta}) \tag{94}$$

Using this expression for Φ in equation (49), we get

$$\frac{v_1}{\left[v_1^2+v_2^2\right]^{1/2}} = \frac{re^{-\theta}F'(re^{-\theta})}{2^{1/2}re^{-\theta}F'(re^{-\theta})}$$

$$\frac{v_2}{\left[v_1^2+v_2^2\right]^{1/2}} = \frac{-re^{-\theta}F'(re^{-\theta})}{2^{1/2}re^{-\theta}F'(re^{-\theta})}$$

Thus

$$\mathbf{v} = (v_1, v_2) = \frac{Q}{2^{1/2}}[\ \mathbf{e}_r - \mathbf{e}_\theta\ ] \tag{95}$$

where $Q^2 = v_1^2 + v_2^2$.

Employing equation (95) in equation (38), the integrability condition for $f\rho$, we obtain

$$\operatorname{curl}\left[\left[-\frac{Q_r}{Q}-\frac{1}{r}+\mu\sigma\frac{Q}{Z^{1/2}}\right]\mathbf{e}_r+\left[-\frac{Q_\theta}{rQ}-\mu\sigma\frac{Q}{Z^{1/2}}\right]\mathbf{e}_\theta\right]=\mathbf{0}$$

This yields the partial differential equation

$$r\,Q_r + Q_\theta + Q = 0 \tag{96}$$

Employing equation (95) in equation (42), the integrability condition for q, we obtain:

$$\operatorname{curl}\left[\left[-\frac{Q_r}{Q}-\frac{1}{r}-\frac{\sigma}{\varepsilon Q Z^{1/2}}\right]\mathbf{e}_r+\left[-\frac{Q_\theta}{rQ}+\frac{\sigma}{Z^{1/2}\varepsilon Q}\right]\mathbf{e}_\theta\right]=\mathbf{0}$$

This yields the partial differential equation

$$rQ_r + Q_\theta - Q = 0 \tag{97}$$

Subtracting equation (97) from equation (96), we get:

$$Q = 0 \tag{98}$$

Using (98) in (95), we get

$$\mathbf{v} = \mathbf{0} \tag{99}$$

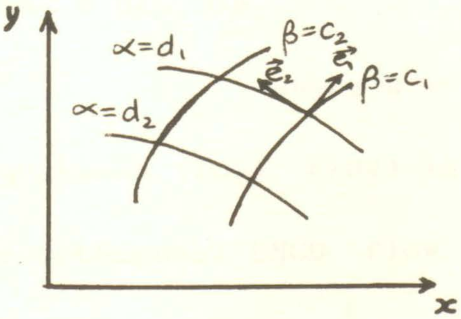
Therefore, we have that the spiral flow pattern for orthogonal EMGD plane steady inviscid flow is not possible.

3.2 PLANE ORTHOGONAL EMGD FLOW WITH ISOMETRIC GEOMETRY

We let

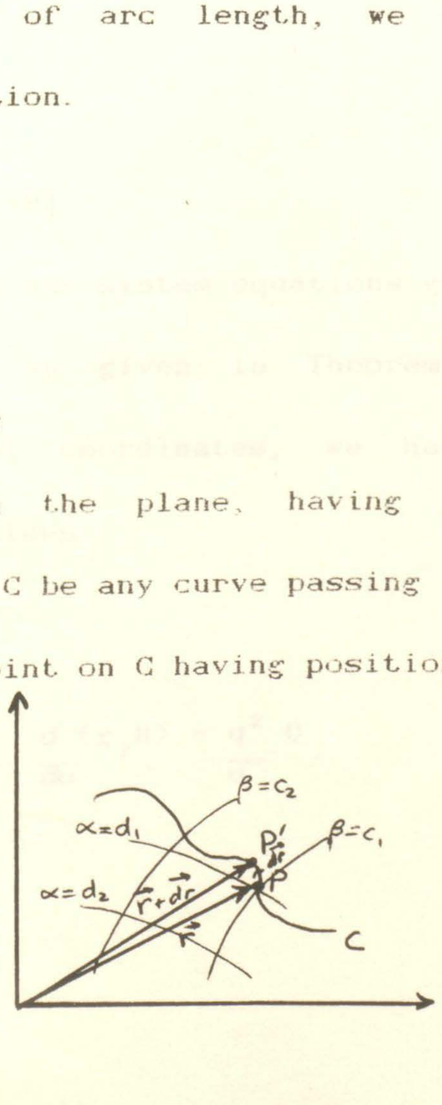
$$x = x(\alpha,\beta) \quad , \quad y = y(\alpha,\beta) \tag{100}$$

define a system of orthogonal curvilinear coordinates in the plane of flow such that the curves $\beta(x,y)$ = constant represent the streamlines and $\alpha(x,y)$ = constant represent their orthogonal trajectories. Letting $\mathbf{e}_1$ be the unit tangent vector to β = const. in the direction of increasing α , $\mathbf{e}_2$ be the unit tangent vector to α = const. and also letting $g_1(\alpha,\beta)d\alpha$ and $g_2(\alpha,\beta)d\beta$ be the components of a vector element of arc length, we have the following pictorial representation.



Now, we take P to be any point in the plane, having position vector $\mathbf{r} = (x(\alpha,\beta), y(\alpha,\beta))$ and let C be any curve passing through P. Letting P′ be a neighbouring point on C having position vector $\mathbf{r}+d\mathbf{r}$, we have that

$$d\mathbf{r} = \mathbf{r}_\alpha d\alpha + \mathbf{r}_\beta d\beta$$



Then, the squared element of arc length is

$$ds^2 = \mathbf{dr} \cdot \mathbf{dr}$$

$$= (\mathbf{r}_\alpha d_\alpha + \mathbf{r}_\beta d\beta) \cdot (\mathbf{r}_\alpha d\alpha + \mathbf{r}_\beta d\beta)$$

$$= |\mathbf{r}_\alpha|^2 \; d\alpha^2 + |\mathbf{r}_\beta| \; d\beta^2$$

Thus, $g_1(\alpha,\beta) = |\mathbf{r}_\alpha|$ and $g_2(\alpha,\beta) = |\mathbf{r}_\beta|$

For this choice of natural net, we get

$$\mathbf{v} = Q(\alpha,\beta)\ \mathbf{e}_1 + 0\mathbf{e}_2 \tag{101}$$

where $Q = |\mathbf{v}|$

and

$$\mathbf{H} = 0\mathbf{e}_1 + H(\alpha,\beta)\mathbf{e}_2 \tag{102}$$

where $H = |\mathbf{H}|$

Employing equations (101), (102) in the system equations governing steady plane orthogonal EMGD flow as given in Theorem 1, and writing the equations in natural coordinates, we have the following system of governing equations

$$\frac{\partial}{\partial\alpha}(\rho g_2 Q)=0 \tag{103}$$

$$\frac{\rho Q}{g_1}\frac{\partial Q}{\partial\alpha} = -\frac{1}{g_1}\frac{\partial p}{\partial\alpha} - \frac{\mu H}{g_1 g_2}\frac{\partial\,(g_2 H)}{\partial\alpha} - \frac{q^2\ Q}{\sigma} \tag{104}$$

$$\frac{\rho Q^2}{g_1} \frac{\partial g_1}{\partial \beta} = \frac{\partial p}{\partial \beta} \tag{105}$$

$$\frac{1}{g_1 g_2} \frac{\partial}{\partial \alpha} (g_2 H) = \mu \sigma Q H \tag{106}$$

$$\frac{\partial}{\partial \beta} (g_1 q Q) = 0 \tag{107}$$

$$\frac{\partial}{\partial \alpha} (g_2 q Q) = (-q \sigma g_1 g_2)/\varepsilon \tag{108}$$

As well, we employ equations (101), (102) in the integrability conditions (38), (42) and (43) to obtain

$$\frac{\partial^2}{\partial \alpha \partial \beta} \left[\ln(g_2/g_1) \right] = \frac{\partial}{\partial \beta} [\mu \sigma g_1 Q] \tag{109}$$

$$\frac{\partial^2}{\partial \alpha \partial \beta} \left[\ln(g_1/g_2) \right] = \frac{\partial}{\partial \beta} \left[\frac{\sigma g_1}{\varepsilon Q} \right] \tag{111}$$

$$c^{-2} \frac{\partial}{\partial \beta} [g_1 Q] + \frac{\partial}{\partial \beta} [g_1/Q] = 0 \tag{112}$$

where $c^2 = 1/\varepsilon\mu$.

Moreover, employing (101) in the definition of ω, ω = curl **v**, yields:

$$\omega = - \frac{1}{g_1 g_2} \frac{\partial}{\partial \beta} (g_1 Q) \mathbf{k} \tag{113}$$

In the present section we enquire what possible solutions for orthogonal flows of an inviscid compressible fluid are possible when the flow streamlines and their orthogonal trajectories form an isometric net. Therefore, we search for solutions when themetric coefficients of the natural, i.e. streamline, coordinates satisfy the condition

$$g_1^2\,(\alpha,\beta) = g_2^2\,(\alpha,\beta) = g^2(\alpha,\beta) \tag{114}$$

Employing condition (114) in equations (103) to (113), we obtain the following equations, valid for an isometric net

$$\frac{\partial}{\partial\alpha}\,(\rho g Q) = 0 \tag{115}$$

$$\frac{\rho Q}{g}\,\frac{\partial Q}{\partial\alpha} = -\frac{1}{g}\,\frac{\partial p}{\partial\alpha} - \frac{\mu H}{g^2}\,\frac{\partial}{\partial\alpha}\,(gH) - (q^2/\sigma)Q \tag{116}$$

$$\frac{\rho Q^2}{g}\,\frac{\partial g}{\partial\beta} = \frac{\partial p}{\partial\beta} \tag{117}$$

$$\frac{1}{g^2}\,\frac{\partial}{\partial\alpha}\,(gH) = QH \tag{118}$$

$$\frac{\partial}{\partial\beta}\,(gqQ) = 0 \tag{119}$$

$$\frac{\partial}{\partial\,\alpha}\,(gqQ) = \frac{-q\sigma g^2}{\varepsilon} \tag{120}$$

$$\frac{\partial}{\partial\beta}\,[\mu\sigma g Q] = 0 \tag{121}$$

$$\frac{\partial}{\partial\beta}\,[(\sigma g)/(\varepsilon Q)] = 0 \tag{122}$$

$$c^{-2}\,\frac{\partial}{\partial\beta}\,[gQ] + \frac{\partial}{\partial\beta}\,[g/Q] = 0 \tag{123}$$

$$\omega = -\frac{1}{g^2}\frac{\partial}{\partial\beta}(gQ)\mathbf{k} \tag{124}$$

Equations (115) to (120) are the governing equations for the flow, while equations (121) to (123) are the integrability conditions and equation (124) is the definition of **w**.

From equations (121), (124) and the above analysis, we have the following theorem:

Theorem 3. If the flow streamlines and their orthogonal trajectories form an isometric net, then the flow under consideration (i.e. steady plane orthogonal EMGD flow of an inviscid compressible fluid) is irrotational.

Since the flow under consideration is irrotational, when an isometric net is taken, therefore we have, from equation (124), that

$$\frac{\partial}{\partial\beta}[gQ] = 0 \tag{125}$$

Employing equation (125) in equation (123), we have

$$\frac{\partial}{\partial\beta}[g/Q]=0 \tag{126}$$

Integrating equation (125) and (126) with respect to β, we get

$$gQ = f_1(\alpha) \tag{127}$$

and $$g/Q = f_2(\alpha) \tag{128}$$

where f_1 and f_2 are arbitrary functions of the argument α.

Multiplying equation (127) and (128), we find that

$$g = [f_1(\alpha)\ f_2(\alpha)]^{1/2}$$

That is,

$$g = g(\alpha) \tag{129}$$

Dividing equation (127) by (128), we find that

$$Q = \left[\frac{f_1(\alpha)}{f_2(\alpha)}\right]^{1/2}$$

that is,

$$Q = Q(\alpha)$$

From equations (129), (130) and the above analysis, we have the following theorem:

Theorem 4. If the flow streamlines and their orthogonal trajectories form an isometric net, then the flow under consideration (i.e. steady plane orthogonal EMGD flow of an inviscid compressible fluid) is irrotational, and furthermore,

(a) the scale factor for the isometric net is

$$g = g(\alpha) \tag{131}$$

(b) the velocity field is constant on orthogonal trajectories, and is given by

$$\mathbf{v} = Q(\alpha)\ \mathbf{e}_1 \tag{132}$$

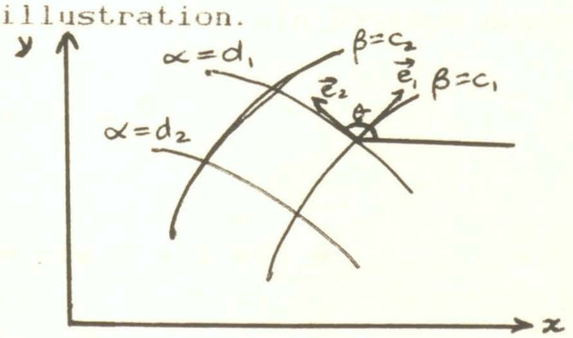
In order to determine what solutions for these orthogonal flows of an inviscid compressible fluid are possible, we use a technique employed by W. Tollmein. Following Tollmein, we introduce the complex variable

$$z = x + iy. \tag{133}$$

Thus,

$$\begin{aligned} z &= z(x,y) \\ &= x(\alpha,\beta) + i\ y(\alpha,\beta) \\ &= z(\alpha,\beta) \end{aligned} \tag{134}$$

The technique employed is understood more clearly by the following illustration.



θ is the angle between $\mathbf{e}_2$ and the positive horizontal direction.

Since, by definition,

$$\mathbf{e}_1 = \frac{z_\alpha}{|z_\alpha|} \quad \text{and} \quad \mathbf{e}_2 = \frac{z_\beta}{|z_\beta|},$$

therefore, $z_\alpha = |z_\alpha|\ \mathbf{e}_1$

and $\quad z_\beta = |z_\beta|\ \mathbf{e}_2$

But

$$\frac{\partial z}{\partial \alpha} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \alpha} = x_\alpha + iy_\alpha$$

and

$$\frac{\partial z}{\partial \beta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \beta} = x_\beta + iy_\beta$$

Therefore,

$$\left|\frac{\partial z}{\partial \alpha}\right| = g \quad , \quad \left|\frac{\partial z}{\partial \beta}\right| = g \tag{135}$$

Also,

$$\begin{aligned} e_1 &= \cos(\Theta - \pi/2) + i\sin(\Theta - \pi/2) \\ &= \cos\Theta\cos\pi/2 + \sin\Theta\sin\pi/2 + i[\sin\Theta\cos\pi/2 \\ &\qquad - \cos\Theta\sin\pi/2] \\ &= (\cos\Theta + i\sin\Theta)(\cos\pi/2 - i\sin\pi/2) \\ &= -i\,e^{i\Theta} \end{aligned}$$

and

$$\begin{aligned} e_2 &= \cos\Theta + i\sin\Theta \\ &= e^{i\Theta} \end{aligned}$$

Hence, we have:

$$\frac{\partial z}{\partial \alpha} = -i\,g\,e^{i\Theta} \quad , \quad \frac{\partial z}{\partial \beta} = g\,e^{i\Theta} \tag{136}$$

where $g = g(\alpha)$, by Theorem 4.

Requiring that $\frac{\partial^2 z}{\partial\alpha\partial\beta} = \frac{\partial^2 z}{\partial\beta\partial\alpha}$, and then equating the real and imaginary parts of the resulting equation, we get

$$g'(\alpha) - g\,\frac{\partial\Theta}{\partial\beta} = 0 \tag{137}$$

$$g(\alpha)\,\frac{\partial\Theta}{\partial\alpha} = 0 \tag{138}$$

From equation (138), we get $\Theta = \Theta(\beta)$. Using this in equation (137), we obtain

$$\Theta'(\beta) = +\,\frac{g'(\alpha)}{g(\alpha)} \tag{139}$$

Observing that the left hand side of equation (139) is a function of β and that the right hand side is a function of α, we can equate both sides to a constant, C_1.

Then,

$$\Theta'(\beta) = C_1$$

and

$$\frac{g'(\alpha)}{g(\alpha)} = C_1$$

Integrating these equations, we get

$$\Theta(\beta) = C_1\beta + C_2 \tag{140}$$

$$g(\alpha) = C_3\exp(C_1\alpha) \tag{141}$$

Employing equation (139) in equation (136), we obtain the following expression for z:

$$z = (+\sin\Theta\,\frac{g^2(\alpha)}{g'(\alpha)} + C) + i(-\cos\Theta\,\frac{g^2(\alpha)}{g'(\alpha)} + D)$$

where $g = g(\alpha)$ and $\Theta = \Theta(\beta)$ are given by equation (140), (141).

Hence,

$$x - C = +\,\frac{g^2(\alpha)}{g'(\alpha)}\sin\Theta$$

and

$$y - D = -\ \frac{g^2(\alpha)}{g'(\alpha)}\ \cos\theta$$

Squaring each of the above two equations and summing, we get

$$(x - c)^2 + (y - D)^2 = +\ \frac{g^4(\alpha)}{[g'(\alpha)]^2}$$

Thus, for a fixed α, we generate a circle of radius $\frac{g^2(\alpha)}{g'(\alpha)}$ centered at the point (C,D).

Hence, the lines α = constant form a set of concentric circles. These are the orthogonal trajectories ; the corresponding streamlines are radial lines.

For the limiting case, $g'(\alpha)=0$, we find form equations (137),(138) that

$$\frac{\partial\theta}{\partial\alpha} = \frac{\partial\theta}{\partial\beta} = 0$$

Thus, θ = const, and so we will have that the family of circles becomes a family of parallel straight lines.

CHAPTER IV

INVISCID INCOMPRESSIBLE EMFD ORTHOGONAL FLOWS WITH NON-ZERO CHARGE DENSITY

4.1 *INTEGRABILITY CONDITIONS FOR ORTHOGONAL FLOWS WITH NON-ZERO CHARGE DENSITY*

In this section, we consider the problem of steady plane electro-magneto-fluid-dynamic flow of an inviscid incompressible fluid, wherein the magnetic field **H** lies in the flow plane and the change density $q(x,y)$ is a non-zero function. By taking the material density ρ to be a constant in the system of equations given in Theorem 1, we obtain the following system of governing equations for the flow problem:

$$\operatorname{div} \mathbf{v} = 0 \tag{142}$$

$$\rho(\mathbf{v}\cdot\operatorname{grad})\mathbf{v} = -\operatorname{grad} p + \mu(\operatorname{curl} \mathbf{H})\times\mathbf{H} - \frac{q^2}{\sigma}\mathbf{v} \tag{143}$$

$$\operatorname{curl} \mathbf{H} = \mu\sigma(\mathbf{v}\times\mathbf{H}) \tag{144}$$

$$\operatorname{curl}(q\mathbf{v}) = \mathbf{0} \tag{145}$$

$$\operatorname{div}(q\mathbf{v}) + \frac{q\sigma}{\varepsilon} = 0 \tag{146}$$

This is a system of six scalar equations in six unknown functions $\rho(x,y)$, $q(x,y)$, $\mathbf{v} = (u(x,y), v(x,y))$ and $\mathbf{H} = (H_1(x,y), H_2(x,y))$.

The magnetic field **H** also satisfies

$$\mathrm{div}\ \mathbf{H} = 0. \tag{147}$$

Using this system of equations, we find the integrability conditions for orthogonal flows of an incompressible fluid with non-zero charge density. Since the flow is orthogonal, we have:

$$\mathbf{H} = \mathbf{k} \times f(x,y)\mathbf{v} \tag{148}$$

where $f(x,y)$ is some scalar function:

The integrability condition for the function $f(x,y)$ is obtained by employing equation (148) in equations (144), (147), while the integrability condition for the $q(x,y)$ is obtained from equations (145), (146).

Employing equation (148) in equation (144), we get

$$\begin{aligned}
\mathbf{0} &= \mathrm{curl}[\mathbf{k} \times f(x,y)\mathbf{v}] - \mu\sigma[\mathbf{v} \times (\mathbf{k} \times f(x,y)\mathbf{v})] \\
&= \mathbf{k}(\mathrm{div}\ f\mathbf{v}) - f\mathbf{v}(\mathrm{div}\ \mathbf{k}) + (f\mathbf{v}\cdot\mathrm{grad})\mathbf{k} - (\mathbf{k}\cdot\mathrm{grad})f\mathbf{v} \\
&\qquad -\mu\sigma[(f\mathbf{v}\cdot\mathbf{v})\mathbf{k} - (\mathbf{v}\cdot\mathbf{k})f\mathbf{v}] \\
&= (\mathrm{div}\ f\mathbf{v})\mathbf{k} - \mu\sigma(f\mathbf{v}\cdot\mathbf{v})\mathbf{k} \\
&= [\mathrm{div}\ f\mathbf{v} - \mu\sigma(f\mathbf{v}\cdot\mathbf{v})]\mathbf{k}
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &= \mathrm{div}\ f\mathbf{v} - \mu\sigma f(\mathbf{v}\cdot\mathbf{v}) \\
&= (\mathrm{grad}\ f)\cdot\mathbf{v} + f\ \mathrm{div}\ \mathbf{v} - \mu\sigma f(\mathbf{v}\cdot\mathbf{v})
\end{aligned}$$

Hence,

$$0 = \mathrm{grad}(\ln f)\cdot\mathbf{v} - \mu\sigma(\mathbf{v}\cdot\mathbf{v}) \tag{149}$$

Employing equation (148) in equation (147), we get

$$0 = \mathrm{div}[\mathbf{k} \times f(x,y)\mathbf{v}]$$

$$= f\mathbf{v}\cdot(\mathrm{curl}\ \mathbf{k}) - \mathbf{k}\cdot(\mathrm{curl}\ f\mathbf{v})$$

$$= -(\mathrm{curl}\ f\mathbf{v})\cdot\mathbf{k}$$

Thus,

$$\mathbf{0} = \mathrm{curl}\ f\mathbf{v}$$

$$= f\ \mathrm{curl}\ \mathbf{v} + (\mathrm{grad}\ f \times \mathbf{v})$$

Hence,

$$\mathbf{0} = \mathrm{grad}(\ln f) \times \mathbf{v} + \mathrm{curl}\ \mathbf{v} \qquad (150)$$

Taking the vector product of equation (150) with $\mathbf{v}$, and using (149) in the resulting equation, we get

$$\mathrm{grad}(\ln f) = \frac{(\mathrm{curl}\ \mathbf{v})\times\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} + \mu\sigma\mathbf{v} \qquad (151)$$

Taking the curl of equation (151), it follows that

$$\mathrm{curl}\left[\frac{(\mathrm{curl}\ \mathbf{v}) \times \mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} + \mu\sigma\mathbf{v}\right] = \mathbf{0} \qquad (152)$$

which is the integrability condition for the function f.

We now examine equations (145) and (146).

Equation (145) is:

$$\mathbf{0} = \mathrm{curl}\ q\mathbf{v}$$

$$= q\ \mathrm{curl}\ \mathbf{v} + (\mathrm{grad}\ q) \times \mathbf{v}$$

Thus, we have

$$\mathbf{0} = \text{curl } \mathbf{v} + \text{grad}(\ln q) \times \mathbf{v} \tag{153}$$

Equation (146) is:

$$\begin{aligned} 0 &= \text{div}(q\mathbf{v}) + (q\sigma)/\varepsilon \\ &= q \text{ div } \mathbf{v} + \text{grad } q \cdot \mathbf{v} + (q\sigma)/\varepsilon \\ &= \text{grad } q \cdot \mathbf{v} + (q\sigma)/\varepsilon \end{aligned}$$

Thus, we have

$$0 = (\text{grad } \ln q) \cdot \mathbf{v} + \sigma/\varepsilon. \tag{154}$$

Taking the vector product of equation (153) with $\mathbf{v}$, and using (154) in the resulting equation, we get

$$\text{grad}(\ln q) = \frac{(\text{curl } \mathbf{v}) \times \mathbf{v} - (\sigma/\varepsilon)\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}. \tag{155}$$

Taking the curl of equation (155), we get the integrability condition for q given by

$$\text{curl}\left[\frac{(\text{curl } \mathbf{v}) \times \mathbf{v} - (\sigma/\varepsilon)\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right] = \mathbf{0}. \tag{156}$$

It follows, from equations (152) and (156), that for plane orthogonal flows, the vector field $\mathbf{v}$ satisfies

$$\text{curl}\left[\varepsilon\mu\mathbf{v} + \mathbf{v}/(\mathbf{v}\cdot\mathbf{v})\right] = \mathbf{0}. \tag{157}$$

Equation (157) results from the subtraction of (156) from (152), and is thus the integrability condition of the quotient function $\frac{f(x,y)}{q(x,y)}$.

Furthermore, we note that any two of the three integrability conditions, (152), (156), (157), are a linearly independent set.

4.2 *PLANE ORTHOGONAL EMFD FLOW WITH ISOMETRIC GEOMETRY*

As in the previous chapter, we let

$$x = x(\alpha,\beta) \quad , \quad y = y(\alpha,\beta)$$

define a system of orthogonal curvilinear coordinates in the plane of flow such that the curves $\beta(x,y)$=constant represent the streamlines and $\alpha(x,y)$=constant represent their orthogonal trajectories. We define $\mathbf{e}_1$ to be the unit tangent vector to β=const. in the direction of increasing α, and $\mathbf{e}_2$ to be the unit tangent vector to α = const. As well, we let $g_1(\alpha,\beta)d\alpha$ and $g_2(\alpha,\beta)d\beta$ be the components of a vector element of arc length. Then, we have:

$$\mathbf{v} = Q(\alpha,\beta)\ \mathbf{e}_1 + 0\ \mathbf{e}_2 \tag{158}$$

$$\mathbf{H} = 0\mathbf{e}_1 + H(\alpha,\beta)\mathbf{e}_2 \tag{159}$$

$$ds^2 = g_1^2(\alpha,\beta)d\alpha^2 + g_2^2(\alpha,\beta)d\beta^2$$

Employing equations (158), (159) in the system of equations governing steady plane orthogonal EMFD flow, given by equations (142) to (146), and writing the equations in natural coordinates, we have the following system of governing equations:

$$\frac{\partial}{\partial \alpha}(g_2 Q) = 0 \tag{160}$$

$$\frac{\rho Q}{g_1}\frac{\partial Q}{\partial \alpha} = \frac{-1}{g_1}\frac{\partial p}{\partial \alpha} - \frac{\mu H}{g_1 g_2}\frac{\partial}{\partial \alpha}(g_2 H) - \frac{q^2 Q}{\sigma} \tag{161}$$

$$\frac{\rho Q^2}{g_1}\frac{\partial g_1}{\partial \beta} = \frac{\partial p}{\partial \beta} \tag{162}$$

$$\frac{1}{g_1 g_2}\frac{\partial}{\partial \alpha}(g_2 H) = \mu \sigma Q H \tag{163}$$

$$\frac{\partial}{\partial \beta}(g_1 q Q) = 0 \tag{164}$$

$$\frac{\partial}{\partial \alpha}(g_2 q Q) = (-q\sigma g_1 g_2)/\varepsilon \tag{165}$$

As well, we employ equations (158), (159) in the integrability conditions (152), (156) and (157) to obtain

$$\frac{\partial^2}{\partial \alpha \partial \beta}(\ln g_1 Q) = -\mu\sigma \frac{\partial}{\partial \beta}(g_1 Q) \tag{166}$$

$$\frac{\partial^2}{\partial \alpha \partial \beta}(\ln g_1 Q) = \frac{\sigma}{\varepsilon}\frac{\partial}{\partial \beta}(g_1 / Q) \tag{167}$$

Hence,

$$\frac{1}{c^2}\frac{\partial}{\partial\beta}(g_1Q) + \frac{\partial}{\partial\beta}(g_1/Q) = 0 \qquad (168)$$

where $c^2 = \dfrac{1}{\varepsilon\mu}$.

In this section we enquire what possible solutions for orthogonal flows of an inviscid incompressible fluid are possible when flow streamlines and their orthogonal trajectories form an isometric net. Therefore, we search for solutions when the metric coefficents of the natural, i.e. streamline, coordinates satisfy the condition

$$g_1(\alpha,\beta) = g_2(\alpha,\beta) = g(\alpha,\beta) \quad - \text{say} - \qquad (169)$$

Employing condition (169) in equations (160) to (168), we obtain the following equations

$$\frac{\partial}{\partial\alpha}(gQ) = 0 \qquad (170)$$

$$\frac{\rho Q}{g}\frac{\partial Q}{\partial\alpha} = \frac{-1}{g}\frac{\partial p}{\partial\alpha} - \frac{\mu H}{g^2}\frac{\partial}{\partial\alpha}(gH) - \frac{q^2 Q}{\sigma} \qquad (171)$$

$$\frac{\rho Q^2}{g}\frac{\partial g}{\partial\beta} = \frac{\partial p}{\partial\beta} \qquad (172)$$

$$\frac{1}{g^2}\frac{\partial}{\partial\alpha}(gH) = QH\mu\sigma \qquad (173)$$

$$\frac{\partial}{\partial\beta}(gqQ) = 0 \qquad (174)$$

$$\frac{\partial}{\partial \alpha}(gqQ) = \frac{-q\sigma g^2}{\varepsilon} \tag{175}$$

$$\frac{\partial^2}{\partial \alpha \partial \beta}(\ln gQ) = -\mu\sigma \frac{\partial}{\partial \beta}(gQ) \tag{176}$$

$$\frac{\partial^2}{\partial \alpha \partial \beta}(\ln gQ) = \frac{\sigma}{\varepsilon} \frac{\partial}{\partial \beta}(g/Q) \tag{177}$$

$$\frac{1}{c^2}\frac{\partial}{\partial \beta}(gQ) + \frac{\partial}{\partial \beta}(g/Q) = 0. \tag{178}$$

Employing equation (170) in equations (176) and (177), we obtain

$$\frac{\partial}{\partial \beta}(gQ) = 0 \tag{179}$$

and

$$\frac{\partial}{\partial \beta}(g/Q) = 0 \tag{180}$$

Integrating equation (179) and (180) with respect to β, and using equation (170) we get

$$gQ = f_1 \tag{181}$$

and

$$g/Q = f_2(\alpha) \tag{182}$$

where f_1 is an arbitrary constant and f_2 is an arbitrary function of the argument α.

From equations (181) and (182), we have that

$$g = \left[f_1 f_2(\alpha)\right]^{1/2} = g(\alpha) \tag{183}$$

and

$$Q = \left[\frac{f_1}{f_2(\alpha)} \right] = Q(\alpha) \tag{184}$$

We use a technique employed by W. Tollmein [ ] in order to determine what solutions for orthogonal flows of an inviscid incompressible fluid are possible. Following Tollmein, we introduce the ordinary complex variable $z = x + iy$.

Thus,

$$\begin{aligned} z &= z(x,y) \\ &= x(\alpha,\beta) + i\ y(\alpha,\beta) \\ &= z(\alpha,\beta) \end{aligned} \tag{185}$$

Since

$$\frac{\partial z}{\partial \alpha} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \alpha} = x_\alpha + iy_\alpha$$

and

$$\frac{\partial z}{\partial \beta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \beta} = x_\beta + iy_\beta$$

therefore

$$\left| \frac{\partial z}{\partial \alpha} \right| = g \qquad , \qquad \left| \frac{\partial z}{\partial \beta} \right| = g \tag{186}$$

Using the fact that the curves α = const. and β = const. form an orthogonal net, we write, as in the compressible case,

$$\frac{\partial z}{\partial \alpha} = -ige^{i\theta} \qquad , \qquad \frac{\partial z}{\partial \beta} = ge^{i\theta} \tag{187}$$

where $g = g(\alpha)$ by equation (183).

Requiring that $z_{\alpha\beta} = z_{\beta\alpha}$, and then equating the real and imaginary parts of the resulting equation, we get

$$\frac{dg}{d\alpha} - g(\alpha)\frac{\partial\theta}{\partial\beta} = 0 \tag{188}$$

$$g(\alpha)\frac{\partial\theta}{\partial\alpha} = 0 \tag{189}$$

From equation (189), we get $\theta = \theta(\beta)$. Using this in equation (188), we obtain

$$\theta'(\beta) = + \frac{g'(\alpha)}{g(\alpha)} \tag{190}$$

We observe that equation (190) equates a function of α to a function of β. Thus, both functions must be constant valued. We write

$$\theta'(\beta) = \frac{g'(\alpha)}{g(\alpha)} = c_1.$$

Thus,

$$\theta(\beta) = c_1\beta + c_2 \tag{191}$$

and

$$g(\alpha) = c_3\exp[\, c_1\alpha\,] \tag{192}$$

where C_1, C_2 and C_3 are constants.

Employing equation (190) in equation (187) and integrating, we obtain the following expression for z:

$$z = \left(+\sin\theta \frac{g^2}{g'(\alpha)} + C\right) + i\left(-\cos\theta \frac{g^2}{g'(\alpha)} + D\right)$$

where $\theta(\beta)$ and $g(\alpha)$ are given by equations (191), (192)

Hence,

$$x - C = +\sin\theta \frac{g^2(\alpha)}{g'(\alpha)}$$

and

$$y - D = -\cos\theta \frac{g^2(\alpha)}{g'(\alpha)}$$

Squaring each of the above two equations and summing we get

$$(x-C)^2 + (y - D)^2 = \frac{g^4(\alpha)}{(g'(\alpha))^2}$$

Taking α = const., we generate a family of concentric circles, centered at (C, D) and of radius $\frac{g^2(\alpha)}{g'(\alpha)}$.

The streamlines corresponding to this family of orthogonal trajectories are radial lines emanating from the point (C,D). For the limiting case $g'(\alpha) = 0$, we find from equations (188), (189) that

$$\frac{\partial\theta}{\partial\alpha} = \frac{\partial\theta}{\partial\beta} = 0$$

Hence, θ = constant, and so we will have that the family of radial streamlines degenerates to a family of parallel straight lines

The above conclusion has been arrived at by analysis employing equations (142), (144) to (147). However, equation (143), the linear momentum equation, has not yet been used. In the following, we verify that the proposed solutions of radial flow and parallel straight flow are indeed valid, by finding the pressure from equation (143) after the other flow variables have been determined. The examples of radial flow and parallel straight flow which should have followed the text are deferred to the next chapter, where the more involved case of viscous, incompressible flow is considered.

CHAPTER V

VISCOUS INCOMPRESSIBLE EMFD ORTHOGONAL FLOWS WITH NON-ZERO CHARGE DENSITY

5.1 *BASIC EQUATIONS*

Steady three dimensional flow of a viscous, incompressible EMFD fluid is governed by the following system of equations. [10]

$$\operatorname{div} \mathbf{v} = 0 \tag{193}$$

$$\rho(\mathbf{v}\circ \operatorname{grad})\mathbf{v} + \operatorname{grad} p = \eta\, \nabla^2\mathbf{v} + \mu\, \mathbf{J} \times \mathbf{H} + q\mathbf{E} \tag{194}$$

$$\mathbf{J} = \mathbf{I} + q\mathbf{v} = \operatorname{curl} \mathbf{H} \tag{195}$$

$$\mathbf{I} = \sigma\, (\mathbf{E} + \mu\, \mathbf{v}\times\mathbf{H}) \tag{196}$$

$$\operatorname{curl} \mathbf{E} = \mathbf{0} \tag{197}$$

$$\operatorname{div} \mathbf{E} = \frac{q}{\varepsilon} \tag{198}$$

This is a system of 14 scalar equations in fourteen unknowns **v** (the velocity field), **H** (the magnetic field), **E** (the electric field), **J** (the current density function), q (the charge density) and p (the pressure). In this system, σ, μ, ε and η are respectively the constant electric conductivity, the constant magnetic permeability, the constant permitivity, and the constant dynamic viscosity. The magnetic field **H** satisfies an additional equation

$$\operatorname{div} \mathbf{H} = 0 \tag{199}$$

expressing the absence of magnetic poles in flow. From equations (195) and (196), we obtain the conduction current density **I** and the electric field **E** given by

$$\mathbf{I} = \text{curl } \mathbf{H} - q\,\mathbf{v} \tag{200}$$

$$\mathbf{E} = \frac{1}{\sigma}\,(\text{curl } \mathbf{H} - q\mathbf{v}) - \mu\,\mathbf{v}\times\mathbf{H} \tag{201}$$

Eliminating **E**, **J** from equations (194), (197) and (198) by using their expressions from (201) and (195), we get

$$\rho(\mathbf{v}\circ\text{grad})\mathbf{v} + \text{grad } p = \eta\nabla^2\mathbf{v} + \mu(\text{curl } \mathbf{H})\times\mathbf{H} + \frac{q}{\sigma}\,(\text{curl } \mathbf{H} - q\mathbf{v}) - \mu q\,\mathbf{v}\times\mathbf{H} \tag{202}$$

$$\text{curl } [\,\sigma^{-1}(\text{curl } \mathbf{H} - q\mathbf{v}) - \mu\,\mathbf{v}\times\mathbf{H}] = 0 \tag{203}$$

$$\text{div } [\sigma^{-1}(\text{curl } \mathbf{H} - q\mathbf{v}) - \mu\,\mathbf{v}\times\mathbf{H}] = \frac{q}{\varepsilon} \tag{204}$$

<u>Plane Flows</u>: We investigate steady plane flows defined as those flows in which the velocity field **v** and the magnetic field **H** lie in a plane defined by the rectangular coordinates x, y and all the flow variables are functions of x, y only.

We initiate the decomposition of the vector equations (202) and (203) into their vector components in the flow plane and their vector component perpendicular to the flow plane. This decomposition yields the following four equations:

$$\rho(\mathbf{v}\circ\text{grad})\mathbf{v} + \text{grad } p = \eta\nabla^2\mathbf{v} + \mu\,(\text{curl } \mathbf{H})\times\mathbf{H} - \frac{q^2}{\sigma}\,\mathbf{v} \qquad (205)$$

$$\frac{q}{\sigma}\,\text{curl } \mathbf{H} - \mu q\ \mathbf{v}\times\mathbf{H} = \mathbf{0} \qquad (206)$$

$$\text{curl}(\text{curl } \mathbf{H}) - \mu\sigma\ \text{curl}(\mathbf{v}\times\mathbf{H}) = \mathbf{0} \qquad (207)$$

$$\text{curl}(q\mathbf{v}) = \mathbf{0} \qquad (208)$$

where equations (205), (206) result from (202) and equations (207), (208) are equivalent to (203).

Employing the assumption that flow variables are functions of x,y only, equation (204) reduces to

$$\text{div } (q\mathbf{v}) + \frac{\sigma q}{\varepsilon} = 0. \qquad (209)$$

From equation (206) and the above analysis, we have the following theorems:

Theorem 5 If the charge density $q(x,y)$ is a non-zero function in a steady plane electro-magneto-fluiddynamic incompressible viscous flow having the magnetic field **H** in the flow plane, then the flow is governed by the system.

$$\text{div } \mathbf{v} = 0 \qquad (210)$$

$$\rho(\mathbf{v}\circ\text{grad})\mathbf{v} + \text{grad } p = \eta\nabla^2\mathbf{v} + \mu\,(\text{curl } \mathbf{H})\times\mathbf{H} - q^2\sigma^{-1}\mathbf{v} \qquad (211)$$

$$\text{curl } \mathbf{H} = \mu\sigma\ \mathbf{v}\times\mathbf{H} \qquad (212)$$

$$\operatorname{curl}(q\mathbf{v}) = \mathbf{0} \tag{213}$$

$$\operatorname{div}(q\mathbf{v}) = -(q\sigma)\varepsilon^{-1}. \tag{214}$$

This is a system of six scalar equations in six unknown functions $\mathbf{v}(x,y) = (u(x,y),\ v(x,y))$, $\mathbf{H} = (H_1(x,y),\ H_2(x,y))$, $p(x,y)$, and $q(x,y)$. The magnetic field $\mathbf{H}$ also satisfies

$$\operatorname{div}\mathbf{H} = 0. \tag{215}$$

Theorem 6 If the charge density $q(x,y)$ is identically zero in a steady plane electro-magneto-gasdynamic flow having the magnetic field $\mathbf{H}$ in the flow plane, then the flow is governed by the system:

$$\operatorname{div}\mathbf{v} = 0$$

$$\rho(\mathbf{v}\cdot\operatorname{grad})\mathbf{v} + \operatorname{grad} p = \eta\nabla^2\mathbf{v} + \mu(\operatorname{curl}\mathbf{H})\times\mathbf{H}$$

$$\operatorname{curl}[\operatorname{curl}\mathbf{H} - \mu\sigma(\mathbf{v}\times\mathbf{H})] = \mathbf{0}.$$

This is a system of five scalar equations in five unknown functions $\mathbf{v} = (u(x,y),\ v(x,y))$, $\mathbf{H} = (H_1(x,y),\ H_2(x,y))$ and $p(x,y)$. The magnetic field $\mathbf{H}$ also satisfies

$$\operatorname{div}\mathbf{H} = 0. \tag{216}$$

5.2 *INTEGRABILITY CONDITIONS FOR ORTHOGONAL FLOWS WITH NON-ZERO CHARGE DENSITY*

From the definition of plane orthogonal flows, when $\mathbf{v}$ and $\mathbf{H}$ are in the flow plane, we have

$$\mathbf{H} = \mathbf{k} \times f(x,y)\mathbf{v} \tag{217}$$

where $f(x,y)$ is a scalar function.

Employing equation (217) in equations (212) and (215), we obtain the integrability condition on $f(x,y)$. Equations (208) and (209) yield the integrability condition on $q(x,y)$. The calculations performed in order to obtain the integrability conditions here are precisely the same as those performed in the case of inviscid flow. Thus, we simply state the results as follows:

$$\text{grad}\,(\ln f) = \frac{(\text{curl}\,\mathbf{v})\times\mathbf{v}}{\mathbf{v}\circ\mathbf{v}} + \mu\sigma\,\mathbf{v} \tag{218}$$

$$\text{grad}\,(\ln q) = \frac{(\text{curl}\,\mathbf{v})\times\mathbf{v} - \sigma\varepsilon^{-1}\mathbf{v}}{\mathbf{v}\circ\mathbf{v}} . \tag{219}$$

Taking the curl of equations (218), (219), we obtain

$$\mathrm{curl}\left[\frac{(\mathrm{curl}\ \mathbf{v})\mathrm{x}\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} + \mu\sigma\mathbf{v}\right] = 0 \tag{220}$$

$$\mathrm{curl}\left[\frac{(\mathrm{curl}\ \mathbf{v})\mathrm{x}\mathbf{v}-\sigma\varepsilon^{-1}\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right] = 0. \tag{221}$$

Subtracting equation (221) from (220), we get the integrability condition on $\frac{f(x,y)}{q(x,y)}$:

$$\mathrm{curl}\left[\mu\sigma\mathbf{v} + \frac{(\sigma/\varepsilon)\ \mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right] = 0 \tag{222}$$

which is rewritten as

$$\mathrm{curl}\left[\varepsilon\mu\mathbf{v} + \frac{\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right] = 0. \tag{223}$$

5.3 *PLANE ORTHOGONAL EMFD VISCOUS FLOW WITH ISOMETRIC GEOMETRY*

As in the previous chapters, we let

i. $x = x(\alpha,\beta)$, $y= y(\alpha,\beta)$ define a system of orthogonal curvilinear coordinates in the plane of flow, so that the curves $\beta(x,y)$ = constant represent the streamlines and $\alpha(x,y)$ = constant represent their orthogonal trajectories.

ii. $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit tangent vectors to β = const. and α = const. , respectively.

iii. $g_1(\alpha,\beta)d\alpha$ and $g_2(\alpha,\beta)d\beta$ be the components of a vector element of arc length.

Then we have

$$\mathbf{v} = Q(\alpha,\beta)\ \mathbf{e}_1 + 0\ \mathbf{e}_2 \tag{224}$$

$$\mathbf{H} = 0\ \mathbf{e}_1 + H(\alpha,\beta)\ \mathbf{e}_2 = \mathbf{k} \times f(\alpha,\beta)\mathbf{v} \tag{225}$$

$$ds^2 = g_1^2(\alpha,\beta)\ d\alpha^2 + g_2^2(\alpha,\beta)\ d\beta^2 \tag{226}$$

where $f(\alpha,\beta)$ is some scalar function

$$Q = |\ \mathbf{v}\ | \tag{227}$$

and

$$H = |\ \mathbf{H}\ |. \tag{228}$$

Employing equations (224), (226) in the system of equations governing steady plane orthogonal EMFD viscous flow, given by equations (210) to (215), and writing the equations in natural coordinates, we have the following governing equations:

$$(g_2Q)_\alpha = 0 \tag{229}$$

$$g_1^{-1}\rho QQ_\alpha + g_1^{-1}p_\alpha = \eta \{ g_2^{-1} [(g_1g_2)^{-1}(g_1Q)_\beta]_\alpha \} - q^2Q\,\sigma^{-1}$$
$$-\mu fQ\,(g_1g_2)^{-1}\,(g_2fQ)_\alpha \tag{230}$$

$$g_2^{-1}\rho QQ_\beta - \rho Q\,(g_1g_2)^{-1}(g_1Q)_\beta + g_2^{-1}p_\beta$$
$$= -\eta\, g_1^{-1}\,[(g_1g_2)^{-1}(g_1Q)_\beta]_\alpha \tag{231}$$

$$(g_1g_2)^{-1}\,(g_2fQ)_\alpha = \mu\sigma fQ^2 \tag{232}$$

$$(g_1qQ)_\beta = 0 \tag{233}$$

$$(g_1g_2)^{-1}\,(g_2qQ)_\alpha = -q\,\sigma\,\varepsilon^{-1} \tag{234}$$

$$(g_1fQ)_\beta = 0 \tag{235}$$

As well, we employ equations (227),(228) in the integrability conditions (221),(225) and (226) to obtain:

$$[\ln(g_1Q)]_{\alpha\beta} = -\,\mu\sigma\,(g_1Q)_\beta \tag{236}$$

$$[\ln(g_1Q)]_{\alpha\beta} = \sigma\,\varepsilon^{-1}\,(Q^{-1}g_1)_\beta \tag{237}$$

$$\frac{1}{c^2}\,(g_1Q)_\beta + (g_1Q^{-1})_\beta = 0 \tag{238}$$

We now enquire what solutions are possible for orthogonal EMFD flows of a viscous incompressible fluid, when the streamlines and their orthogonal trajectories form an isometric net. That is, we search for solutions when the metric coefficients of the streamline coordinates satisfy the condition

$$g_1(\alpha,\beta) = g_2(\alpha,\beta) = g(\alpha,\beta) \text{ -say-} \tag{239}$$

Employing condition (239) in equations (229) to (238), we obtain the following equations

$$(gQ)_\alpha = 0 \tag{240}$$

$$\rho g^{-1} Q\, Q_\alpha + g^{-1} p_\alpha = \eta\, g^{-1}[g^{-2}(gQ)_\beta]_\beta - \mu f Q\, g^{-2}\, (gfQ)_\alpha - q^2 \sigma^{-1} Q \tag{241}$$

$$\rho g^{-1}\, Q\, Q_\beta - \rho\, g^{-2}\, Q\, (Qg)_\beta + g^{-1} p_\beta = -\eta\, g^{-1}[g^{-2}(gQ)_\beta]_\alpha \tag{242}$$

$$g^{-2}\, (gfQ)_\alpha = f\, Q^2\, \mu\sigma \tag{243}$$

$$(g\, q\, Q)_\beta = 0 \tag{244}$$

$$(g\, q\, Q)_\alpha = -\, \sigma\, \varepsilon^{-1}\, q\, g^2 \tag{245}$$

$$(g\, f\, Q)_\beta = 0 \tag{246}$$

$$[\ln(gQ)]_{\alpha\beta} = -\mu\sigma\, (gQ)_\beta \tag{247}$$

$$[\ln(gQ)]_{\alpha\beta} = \sigma\, \varepsilon^{-1}\, (g\, Q^{-1})_\beta \tag{248}$$

$$\varepsilon^{-2}\, (g\, Q)_\beta + (g\, Q^{-1})_\beta = 0 \tag{249}$$

Employing equation (240) in equations (247) and (248), we get:

$$(g\, Q)_\beta = 0 \tag{250}$$

$$(g\, Q^{-1})_\beta = 0 \tag{251}$$

From (250) and (251), we have that

$$g = g(\alpha) \tag{252a}$$

and $$Q = Q(\alpha). \tag{252b}$$

Following the complex variable approach used by Tollmein [9], we can find the possible flow geometries for this case in precisely the same manner as was done for the inviscid case. In fact, the calculations are precisely the same as in the case of the inviscid, incompressible fluid since we have in both cases that the metric coefficient is a function of α only. Thus, we omit the calculations, and simply state the conclusion that the possible flow geometries are: (i) streamlines are radial lines and (ii) streamlines are straight parallel lines. We note that this conclusion is obtained by studying all equations except for the linear momentum equation. We must verify that these solutions are indeed valid by employing the linear momentum equation to obtain the pressure function, when the other flow variables have been determined. This analysis is carried out in the following.

<u>Radial Flows</u>

We take θ = const. and r = const. to be the families of streamlines and their orthogonal trajectories. Thus, we have

$$\mathbf{v} = Q\,\mathbf{e}_r + 0\,\mathbf{e}_\theta \tag{253}$$

$$\mathbf{H} = 0\ \mathbf{e}_r + H\ \mathbf{e}_\theta \tag{254}$$

Employing equation (253) in equation (218), we get:

$$\mathrm{grad}(\ln f) = \mu\ \sigma\ Q\ \mathbf{e}_r - r^{-1}\ (\ln Q)_\theta\ \mathbf{e}_\theta. \tag{255}$$

Thus, the integrability condition on $f(r,\theta)$ is obtained by taking the curl of both sides of equation (255), and this yields

$$(\ln Q)_{r\theta} = -\ \mu\ \sigma\ (Q)_\theta. \tag{256}$$

Likewise, employing equation (253) in equation (217) yields:

$$\mathrm{grad}\ (\ln q) = -\sigma\ (\varepsilon\ Q)^{-1}\ \mathbf{e}_r - r^{-1}\ (\ln Q)_\theta\ \mathbf{e}_\theta \tag{257}$$

and taking the curl of this equation yields the integrability condition on $q(r,\theta)$, which may be expressed as:

$$(\ln Q)_{r\theta} + \sigma\ (\varepsilon\ Q^2)^{-1}\ Q_\theta = 0. \tag{258}$$

From equations (256) and (258), we have

$$\sigma\ (\varepsilon\ Q^2)\ Q_\theta = \mu\ \sigma\ Q_\theta.$$

Since the electrical conductivity, σ, is a non-zero quantity, thus we may write the above equation as

$$[(\varepsilon\ Q^2)^{-1} - \mu]\ Q_\theta = 0. \tag{259}$$

From equation (259), it follows that either Q is a function of r only or Q is a constant ,of value $(\varepsilon\mu)^{-.5}$, throughout the flow region. The remaining analysis of the flow is divided to accomodate consideration of both of these cases.

Case 1 $[Q = Q(r)]$

Taking $Q = Q(r)$ in equation (255) yields

$$(\ln f)_r = \mu\,\sigma\,Q(r) \tag{260}$$

$$(\ln f)_\theta = 0 \tag{261}$$

Integrating equations (261) and (260) with respect to θ and r, respectively, we get

$$f = F_1(r) \tag{262}$$

where F_1 is an arbitrary function of r, and

$$f = \exp\left[\int \mu\,\sigma\,Q(r)\,dr\right]. \tag{263}$$

Taking $Q = Q(r)$ in equation (257) yields

$$(\ln q)_r = -\,\sigma\,[\varepsilon\,Q(r)]^{-1} \tag{264}$$

$$(\ln q)_\theta = 0. \tag{265}$$

Integrating equations (265) and (264) with respect to θ and r, respectively, we get

$$q = F_2(r) \tag{266}$$

where F_2 is an arbitrary function of r only

and

$$q = \exp\left[\,\sigma/\varepsilon \int Q^{-1}(r)\,dr\,\right]. \tag{267}$$

We compute the magnetic field vector from equation (217), and obtain

$$\mathbf{H} = \{ Q(r) \exp [\mu\sigma \int Q(r)\, dr] \}\, \mathbf{e}_\theta \tag{268}$$

Using the above information in the linear momentum equations, we get

$$p_r = - \rho Q \frac{dQ}{dr} - \frac{\mu f Q}{r} \frac{d}{dr}(r f Q) - \frac{q^2 Q}{\sigma} \tag{269}$$

$$p_\theta = 0. \tag{270}$$

Equation (270) places the constraint on the pressure function that

$$p = p(r). \tag{271}$$

Furthermore, the requirement on p, $\frac{\partial^2 p}{\partial r \partial \theta} = \frac{\partial^2 p}{\partial \theta \partial r}$, is satisfied identically by equations (269) and (270).

The case in which Q is a constant value throughout the flow region may be viewed as a special case of the case $Q = Q(r)$.

Parallel Flows

We take $ax + by =$ const. and $ay - bx =$ const. to be the families of streamlines and their orthogonal trajectories. Taking $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$, equation (223) yields:

$$\mathbf{0} = \text{curl}\, [(\varepsilon\mu\, u + uQ^{-2})\, \mathbf{i} + (\varepsilon\mu\, v + vQ^{-2})\, \mathbf{j}\,]$$

where $Q^2 = u^2 + v^2$.

Thus,

$$[\ \varepsilon\mu\ v + (v/Q^2)\ ]_x - [\ \varepsilon\mu\ u + (u/Q^2)\ ]_y = 0$$

The above equation implies the existence of a function ϕ satisfying

$$\phi_y = \varepsilon\mu\ v + v/Q^2$$

$$\phi_x = \varepsilon\mu\ u + u/Q^2.$$

Thus,

$$u = (\phi_x)/(\varepsilon\mu + Q^{-2})$$

and $$v = (\phi_y)/(\varepsilon\mu + Q^{-2})$$

But

$$[\ \varepsilon\mu + Q^{-2}] = [(\phi_x^2 + \phi_y^2)/(u^2 + v^2)]^{1/2}$$

$$= [(\phi_x^2 + \phi_y^2)^{1/2}]\ /Q$$

Therefore,

$$u = (\phi_x Q)/|\text{grad}\ \underline{\phi}|$$

$$v = (\phi_y Q)/|\text{grad}\ \underline{\phi}|$$

where $|\text{grad}\ \boldsymbol{\phi}| = (\phi_x^2 + \phi_y^2)^{1/2}$.

Taking

$$\phi = F(ay-bx), \tag{276}$$

we get:

$$\mathbf{v} = [Q/(a^2 + b^2)^{1/2}]\ (-b\ \mathbf{i} + a\ \mathbf{j}). \tag{277}$$

Employing equation (277) in the continuity equation, (211), yields

$$Q = Q(ax + by). \tag{278}$$

Employing equations (277) and (278) in the integrability conditions for the functions $f(x,y)$ and $q(x,y)$, we obtain the following expressions for these functions:

$$f(x,y) = L \exp [(\mu\sigma k)(ay-bx)/(a^2+b^2)^{1/2}] \tag{279}$$

and

$$q(x,y) = M \exp [(\sigma k^{-1}/\varepsilon^{-1})(bx-ay)/(a^2+b^2)^{1/2}] \tag{280}$$

where L, M and k are arbitrary constants.

We compute the magnetic field vector from equation (217), and obtain

$$\mathbf{H} = -[(a Q f)/(a^2+b^2)^{1/2}]\, \mathbf{i} - [(b Q f)/(a^2+b^2)^{1/2}]\, \mathbf{j} \tag{281}$$

For convenience, we take

$$m = ay - bx$$

and

$$n = ax + by$$

Then, for the flow under consideration, we have

$$Q = Q(n)$$

$$f = \exp [(\mu\sigma k m)/(a^2+b^2)^{1/2}]$$

$$q = \exp [(-\sigma k \quad m/\varepsilon)/(a + b \quad) \qquad]$$

$$\mathbf{H} = - \{ [a\, Q(n)\, f(m)]/[(a^2+b^2)^{1/2}] [a\mathbf{i} + b\mathbf{j}]$$

The linear momentum equations, expressed in terms of independent variables m and n , are:

$$\rho Q\, Q_m + p_m = \eta\, Q_{nn} - \mu\, f(m)\, Q(n)\, (fQ)_m - (q^2 Q)/\sigma$$

$$\rho Q\, Q_n - \rho Q\, Q_n + p_n = -\eta\, Q_{mn}$$

That is,

$$p_m = \eta\, Q_{nn} - \mu\, f(m)\, Q^2(n)\, f'(m) - (q^2 Q)/\sigma$$

& $$p_n = 0$$

From the above two equations, we have that $p = p(m)$, that is, the pressure function, $p = p(ay-bx)$. As well, the required condition, $p_{mn} = p_{nm}$ is satisfied by the linear momentum equations.

CHAPTER VI

INVISCID COMPRESSIBLE EMGD CONSTANTLY-INCLINED FLOWS WITH NON-ZERO CHARGE DENSITY

6.1 *INTEGRABILITY CONDITIONS AND CIRCULATION FOR CONSTANTLY INCLINED FLOWS WITH NON-ZERO CHARGE DENSITY*

If the charge density $q(x,y)$ is a non-zero function in a steady plane EMGD flow, we have, by Theorem I, the following system of governing equations:

$$\operatorname{div}(\rho \mathbf{v}) = 0 \tag{285}$$

$$\rho(\mathbf{v}\circ\operatorname{grad})\mathbf{v} + \operatorname{grad} p = \mu\,(\operatorname{curl} \mathbf{H})\times\mathbf{H} - \frac{q^2}{\sigma}\mathbf{v} \tag{286}$$

$$\operatorname{curl} \mathbf{H} = \mu\sigma\,(\mathbf{v}\times\mathbf{H}) \tag{287}$$

$$\operatorname{curl}(q\mathbf{v}) = \mathbf{0} \tag{288}$$

$$\operatorname{div}(q\mathbf{v}) + \frac{\sigma q}{\varepsilon} = 0. \tag{289}$$

As well, the magnetic field $\mathbf{H}$ satisfies

$$\operatorname{div} \mathbf{H} = 0. \tag{290}$$

Plane constantly-inclined flows are qualitatively understood as those flows in which the velocity field $\mathbf{v}$ and the magnetic field $\mathbf{H}$ lie in the (x-y) plane, and furthermore, $\mathbf{v}$ and $\mathbf{H}$ are separated by a constant angle. Formally, we define plane constantly-inclined constantly inclined flows as those flows for which the magnetic field and the velocity fied satisfy:

$$\mathbf{H} = \rho\, f_1\, \mathbf{v} + (\mathbf{k} \times f_2\, \rho\, \mathbf{v}) \tag{291}$$

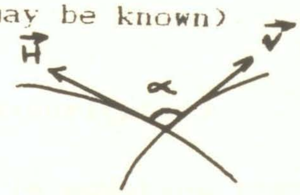
where f_1 and f_2 are functions of x and y.

From equation (291), we compute the following:

$$\mathbf{H} \circ \mathbf{v} = \rho\, f_1\, v^2 \tag{292}$$

$$(\mathbf{v} \times \mathbf{H}) \circ \mathbf{k} = H\, v \sin\alpha = +\rho\, f_2 v^2 \tag{293}$$

(We fix the orientation of the vectors **v** and **H**, as illustrated so that the direction of **k** may be known)



Taking α to be the constant angle of separation between **H** and **v**, the above two equations may be written as

$$H\, v \cos\alpha = \rho\, f_1\, v^2$$

$$H\, v \sin\alpha = -\,\rho\, f_2\, v^2$$

Dividing equation (293) by (292), we get

$$\tan\alpha = +\, f_2\, [f_1]^{-1} \tag{294}$$

Using equation (294) in (291), we have

$$\mathbf{H} = \rho\, f_1\, \mathbf{v} + (\mathbf{k} \times \rho\, f_1 \tan\alpha\, \mathbf{v}) \tag{295}$$

We now employ equation (295) in equations (287) and (290) to find the integrability condition for the product of scalar functions $\rho(x,y)$ and $f_1(x,y)$. Henceforth, $f_1(x,y)$ shall be written as $f(x,y)$

for simplicity.

Employing equation (295) in equation (290) yields:

$$\operatorname{div}(\rho f\mathbf{v} + (\mathbf{k} \times c\rho f\mathbf{v})) = 0 \qquad \text{where } c = \tan\alpha.$$

Thus,

$$0 = \operatorname{div}(\rho f\mathbf{v}) + \operatorname{div}(\mathbf{k} \times c\rho f\mathbf{v})$$

$$= \operatorname{div}(\rho f\mathbf{v})\ c[(\rho f\mathbf{v})\circ\operatorname{curl}\mathbf{k} - \mathbf{k}\circ\operatorname{curl}(\rho f\mathbf{v})]$$

But $\mathbf{k} = (0,0,1)$ is a constant vector, and so curl $\mathbf{k} = \mathbf{0}$.

Thus, we have

$$0 = \operatorname{div}(\rho f\mathbf{v}) - c(\mathbf{k}\circ\operatorname{curl}(\rho f\mathbf{v})) \tag{296}$$

Employing equation (295) in equation (287) yields:

$$\operatorname{curl}(\rho f\mathbf{v} + (\mathbf{k} \times c\rho f\mathbf{v}) = \mu\sigma\mathbf{v} \times [\rho f\mathbf{v} + (\mathbf{k} \times c\rho f\mathbf{v})]$$

Since $\mathbf{v} \times \mathbf{v} = \mathbf{0}$, we write:

$$0 = \operatorname{curl}(\rho f\mathbf{v}) + \operatorname{curl}(\mathbf{k} \times c\rho f\mathbf{v}) - \mu\sigma\mathbf{v} \times (\mathbf{k} \times c\rho f\mathbf{v})$$

$$= \operatorname{curl}(\rho f\mathbf{v}) + c[\ \mathbf{k}(\operatorname{div}(\rho f\mathbf{v}) - \rho f\mathbf{v}(\operatorname{div}\mathbf{k})$$

$$+ (\rho f\mathbf{v}\circ\nabla)\mathbf{k} - (\mathbf{k}\circ\nabla)f\rho\mathbf{v}\ ] - \mu\sigma c\rho f|\mathbf{v}|^2\ \mathbf{k}$$

$$= \operatorname{curl}(\rho f\mathbf{v}) + c\mathbf{k}(\operatorname{div}\rho f\mathbf{v}) - \mu\sigma c\rho f|\mathbf{v}|^2 k$$

Thus,

$$0 = \operatorname{curl}(\rho f\mathbf{v})\circ\mathbf{k} + c\ \operatorname{div}(\rho f\mathbf{v}) - \mu\sigma c\rho f|\mathbf{v}|^2 \tag{297}$$

From (296), we have:

$$\mathbf{k}\circ\operatorname{curl}(\rho f\mathbf{v}) = \frac{1}{c}\operatorname{div}(\rho f\mathbf{v})$$

Employing the above in (297), we get

$$\left(\frac{1}{c} + c\right) \operatorname{div}(\rho f \mathbf{v}) = \mu\sigma c \rho f |\mathbf{v}|^2$$

That is,

$$(c^2 + 1) \operatorname{div}(\rho f \mathbf{v}) = \mu\sigma c^2 \rho f |\mathbf{v}|^2$$

or

$$\operatorname{div}(\rho f \mathbf{v}) = \frac{c^2}{1 + c^2} \mu\sigma f \rho |\mathbf{v}|^2 \tag{298}$$

From (297), we have:

$$\operatorname{div}(\rho f \mathbf{v}) = \mu\sigma\rho f |\mathbf{v}|^2 - \frac{1}{c} \operatorname{curl}(\rho f \mathbf{v}) \circ \mathbf{k}$$

Employing the above in (296), we get

$$\left(c + \frac{1}{c}\right)(\mathbf{k} \circ \operatorname{curl}(\rho f \mathbf{v})) = \mu\sigma\rho f |\mathbf{v}|^2$$

That is,

$$(c^2 + 1)(\mathbf{k} \circ \operatorname{curl}(\rho f \mathbf{v})) = \mu\sigma c \rho f |\mathbf{v}|^2$$

or

$$\operatorname{curl}(\rho f \mathbf{v}) = \frac{c}{1 + c^2} \mu\sigma\rho f |\mathbf{v}|^2 \mathbf{k} \tag{299}$$

We rewrite equation (298) as:

$$(\nabla f \rho) \circ \mathbf{v} + f\rho \operatorname{div}\mathbf{v} = \frac{c^2}{1 + c^2} \mu\sigma f \sigma |\mathbf{v}|^2$$

Dividing the above equation by $f\rho$, we get:

$$\nabla(\ln f\rho) \circ \mathbf{v} = -\operatorname{div}\mathbf{v} + \frac{c^2}{1 + c^2} \mu\sigma |\mathbf{v}|^2 \tag{300}$$

Likewise, we rewrite equation (299) as:

$$[\nabla(f\rho)\times\mathbf{v} + f\rho\,\mathrm{curl}\,\mathbf{v}] = \frac{c}{1+c^2}\mu\sigma|\mathbf{v}|^2 f\rho\mathbf{k}$$

Dividing the above equaiton by $f\rho$, we get:

$$\nabla(\ln(f\rho))\times\mathbf{v} + \mathrm{curl}\,\mathbf{v}] = \frac{c}{1+c^2}\mu\sigma|\mathbf{v}|^2\mathbf{k}$$

Taking the vector product of the above equation with $\mathbf{v}$ (on the left), we get:

$$\nabla(\ln(f\rho)) = \frac{(\nabla\times\nabla)\times\mathbf{v} + (\mathrm{grad}(\ln f\rho)\circ\mathbf{v})\mathbf{v} - \frac{c}{c^2+1}\mu\sigma|\mathbf{v}|^2\mathbf{k}\times\mathbf{v}}{\mathbf{v}\circ\mathbf{v}}$$

Employing (300) in the above equation, we get:

$$\nabla(\ln f\rho) = \frac{\mathbf{w}\times\mathbf{v} - (\mathrm{div}\,\mathbf{v})\mathbf{v} + \frac{c^2}{1+c^2}\mu\sigma|\mathbf{v}|^2\mathbf{v} + \frac{c}{1+c^2}\mu\sigma|\mathbf{v}|^2\mathbf{v}\times\mathbf{k}}{\mathbf{v}\circ\mathbf{v}}$$

.......(301)

Taking the curl of the above equation, we get:

$$\mathrm{curl}\left[\frac{\omega\times\mathbf{v} - (\mathrm{div}\,\mathbf{v})\mathbf{v} + \frac{c^2}{1+c^2}\mu\sigma|\mathbf{v}|^2\mathbf{v} + \frac{c}{1+c^2}\mu\sigma|\mathbf{v}|^2\mathbf{v}\times\mathbf{k}}{\mathbf{v}\circ\mathbf{v}}\right] = \mathbf{0}$$

.......(302)

This is the integrability condition for the product function $f(x,y)\,\rho(x,y)$.

Equations (288) and (289) yield the integrability conditions for the scalar function $q(x,y)$, which is the same as for the previous cases considered:

$$\mathrm{grad}(\ln(q)) = \frac{(\mathrm{curl}\,\mathbf{v})\times\mathbf{v} - (\mathrm{div}\,\mathbf{v} + \frac{\sigma}{\varepsilon})\mathbf{v}}{\mathbf{v}\circ\mathbf{v}}$$

and so

$$\mathrm{curl}\left[\frac{\omega\times\mathbf{v} - (\mathrm{div}\,\mathbf{v} + \frac{\sigma}{\varepsilon})\mathbf{v}}{\mathbf{v}\circ\mathbf{v}}\right] = 0 \qquad \textbf{(303)}$$

Subtracting equation (303) from equation (302), we obtain the integrability condition for $\frac{\rho f}{q}$:

$$\mathrm{curl}\left[\frac{\frac{c}{1+c^2}\mu\sigma|\mathbf{v}|^2\mathbf{v}\times\mathbf{k} + \frac{c^2}{1+c^2}\mu\sigma|\mathbf{v}|^2\mathbf{v} + \frac{\sigma}{\varepsilon}\mathbf{v}}{\mathbf{v}\circ\mathbf{v}}\right] = 0 \qquad (304)$$

Any two of the three integrability conditions given in equations (302), (303) and (304) are linearly independent.

<u>Circulation</u> We let C be an arbitrary closed curve which moves with the fluid and bounds an area A. By definition of circulation Γ around C, we have that

$$\Gamma = \oint_c \mathbf{v}\circ d\mathbf{r}$$ where $d\mathbf{r}$ is an element of of arc length along c.

As well,

$$\frac{D}{Dt}\Gamma = \oint \mathbf{a}\circ d\mathbf{r}$$

By Stokes Theorem, we obtain:

$$\frac{D}{Dt}\Gamma = \iint_A (\mathrm{curl}\,\mathbf{a})\circ d\mathbf{s}$$

$$= \iint_A (\mathrm{curl}(\mathbf{w}\times\mathbf{v}))\circ d\mathbf{s}$$

We observe that circulation will be conserved, that is $\frac{D\Gamma}{Dt} = 0$, if and only if

$$\mathrm{curl}(\mathbf{w}\times\mathbf{v}) = \mathbf{0}$$

The objective of this section is to determine the flow conditions under which circulation is conserved for an inviscid, incompressible EMFD constantly-inclined flow.

Since the fluid under consideration is incompressible, div $\mathbf{v} = 0$ and equation (302) may be written as follows:

$$\frac{c}{1 + c^2}\,\mu\sigma\,\mathrm{curl}(\mathbf{v}\times\mathbf{k}) + \mathrm{curl}\left(\frac{\mathbf{w}\times\mathbf{v}}{\mathbf{v}\circ\mathbf{v}}\right) + \frac{c^2}{1 + c^2}\,\mu\sigma\,\mathrm{curl}\,\mathbf{v} = \mathbf{0}.$$

But, we note that

i. $\mathrm{curl}(\mathbf{v}\times\mathbf{k}) = \mathbf{v}(\mathrm{div}\,\mathbf{k}) - \mathbf{k}(\mathrm{div}\,\mathbf{v}) + (\mathbf{k}\circ\nabla)\mathbf{v} - (\mathbf{v}\circ\nabla)\mathbf{k}$

$$= \mathbf{0}$$

ii. $\mathrm{curl}\left(\frac{\mathbf{w}\times\mathbf{v}}{\mathbf{v}\circ\mathbf{v}}\right) = \frac{-2}{Q^3}\nabla Q \times (\mathbf{w}\times\mathbf{v}) + \frac{1}{Q^2}\mathrm{curl}(\mathbf{w}\times\mathbf{v})$

where $Q = |\mathbf{v}|$ and $\omega = \mathrm{curl}\ \mathbf{v}$

Therefore, we have:

$$\mathbf{0} = \frac{-2}{Q^3}\nabla Q \times (\mathbf{w}\times\mathbf{v}) + \frac{1}{Q^2}\mathrm{curl}(\mathbf{w}\times\mathbf{v}) + \frac{c^2}{1 + c^2}\,\mu\sigma\mathbf{w}$$

$$= \frac{-2}{Q^3}\Big[((\nabla Q)\circ\mathbf{v})\mathbf{w} - (\nabla Q\circ\mathbf{w})\mathbf{v}\Big] + \frac{1}{Q^2}\mathrm{curl}\,(\mathbf{w}\times\mathbf{v}) + \frac{c^2}{1 + c^2}\,\mu\sigma\mathbf{w}$$

$$= \frac{-2}{Q^3}((\nabla Q)\circ \mathbf{v})\mathbf{w} + \frac{1}{Q^2}\,\mathrm{curl}\,(\mathbf{w}\times\mathbf{v}) + \frac{c^2}{1+c^2}\,\mu\sigma\mathbf{w}$$

$$= \left[\frac{-2}{Q^3}(\nabla Q\circ \mathbf{v}) + \frac{c^2}{1+c^2}\,\mu\sigma\right]\mathbf{w} + \frac{1}{Q^2}\,\mathrm{curl}\,(\mathbf{w}\times\mathbf{v})$$

Taking the flow under consideration to be such that circulation is conserved, we will have $\mathrm{curl}(\mathbf{w} \times \mathbf{v}) = \mathbf{0}$. Thus, the following equation must be satisfied:

$$\mathbf{w}\left[\frac{-2}{Q^3}(\nabla Q\circ \mathbf{v}) + \frac{c^2}{1+c^2}\,\mu\sigma\right] = \mathbf{0} \tag{305}$$

By equation (305), we may conclude that in a flow which is circulation-preserving, either:

(i) $\omega = \mathbf{0}$, that is, the flow is irrotational (306a)

or

(ii) $\mathrm{grad}\ Q \circ \mathbf{v} = \dfrac{\mu\,\sigma\,c^2}{2(1+c^2)}\,Q^3$. (306b)

We consider two special cases of the conclusions of equation (306).

If the flow is aligned, that is, the velocity field $\mathbf{v}$ and the magnetic field $\mathbf{H}$ lie in the (x,y) plane, and furthermore, $\mathbf{v}$ and $\mathbf{H}$ are everywhere parallel, we have that $c = \tan\theta = 0$. Thus, for aligned flows in which the circulation is preserved, either

(i) $\omega = \mathbf{0}$,

or (ii) $(\mathrm{grad}\ Q) \circ \mathbf{v} = 0$

This conclusion agrees with the result obtained by Chandna, Toews and Prabaharan [4].

If the flow is orthogonal, then $c = \tan 90^\circ = \infty$. Thus, for orthogonal flows in which the circulation is preserved, either

$$\text{(i)} \quad \omega = 0$$

$$\text{or} \qquad \text{(ii)} \quad (\text{grad } Q) \circ \mathbf{v} = \frac{\mu\sigma}{2} Q^3$$

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